Ordinary Differential Equations

Lecture Notes

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August 18, 2025

1 Lecture 6: Stability Theory and Lyapunov Methods

1.0.1 Introduction to Stability Theory

Stability theory addresses one of the most fundamental questions in dynamical systems: given a solution to a differential equation, what happens to nearby solutions? This question is crucial for understanding the robustness of system behavior and predicting long-term dynamics. While linearization provides local stability information near equilibria, Lyapunov theory offers global methods that can analyze stability over large regions of phase space.

The concept of stability has profound practical implications. In engineering, we need to ensure that control systems remain stable under perturbations. In ecology, we want to understand whether population equilibria can persist under environmental fluctuations. In economics, stability analysis helps predict whether market equilibria are robust to external shocks.

Stability theory provides rigorous mathematical frameworks for addressing these questions. The methods developed by Aleksandr Lyapunov in the late 19th century remain the cornerstone of modern stability analysis, offering both theoretical insights and practical tools for system design and analysis.

1.0.2 Types of Stability

Stability comes in several forms, each capturing different aspects of system behavior under perturbations. Understanding these distinctions is crucial for applying the appropriate analytical tools.

Lyapunov Stability: A solution $\mathbf{x}(t)$ is Lyapunov stable if solutions starting near $\mathbf{x}(0)$ remain near $\mathbf{x}(t)$ for all future times. Formally, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|\mathbf{x}_0 - \mathbf{x}(0)| < \delta$, then $|\mathbf{x}(t; \mathbf{x}_0) - \mathbf{x}(t)| < \epsilon$ for all $t \geq 0$.

Asymptotic Stability: A solution is asymptotically stable if it is Lyapunov stable and nearby solutions actually converge to it as $t \to \infty$. This requires $\lim_{t\to\infty} |\mathbf{x}(t; \mathbf{x}_0) - \mathbf{x}(t)| = 0$ for initial conditions sufficiently close to $\mathbf{x}(0)$.

Exponential Stability: The strongest form of stability, where nearby solutions converge exponentially fast. There exist constants $M > 0$ and $\alpha > 0$ such that $|\mathbf{x}(t; \mathbf{x}_0) - \mathbf{x}(t)| \leq M|\mathbf{x}_0 - \mathbf{x}(0)|e^{-\alpha t}$.

Global Stability: When stability properties hold for all initial conditions in the phase space, not just those in a neighborhood of the reference solution.

For autonomous systems, we typically focus on the stability of equilibrium points, where the reference solution is constant: $\mathbf{x}(t) = \mathbf{x}^*$ for all t.

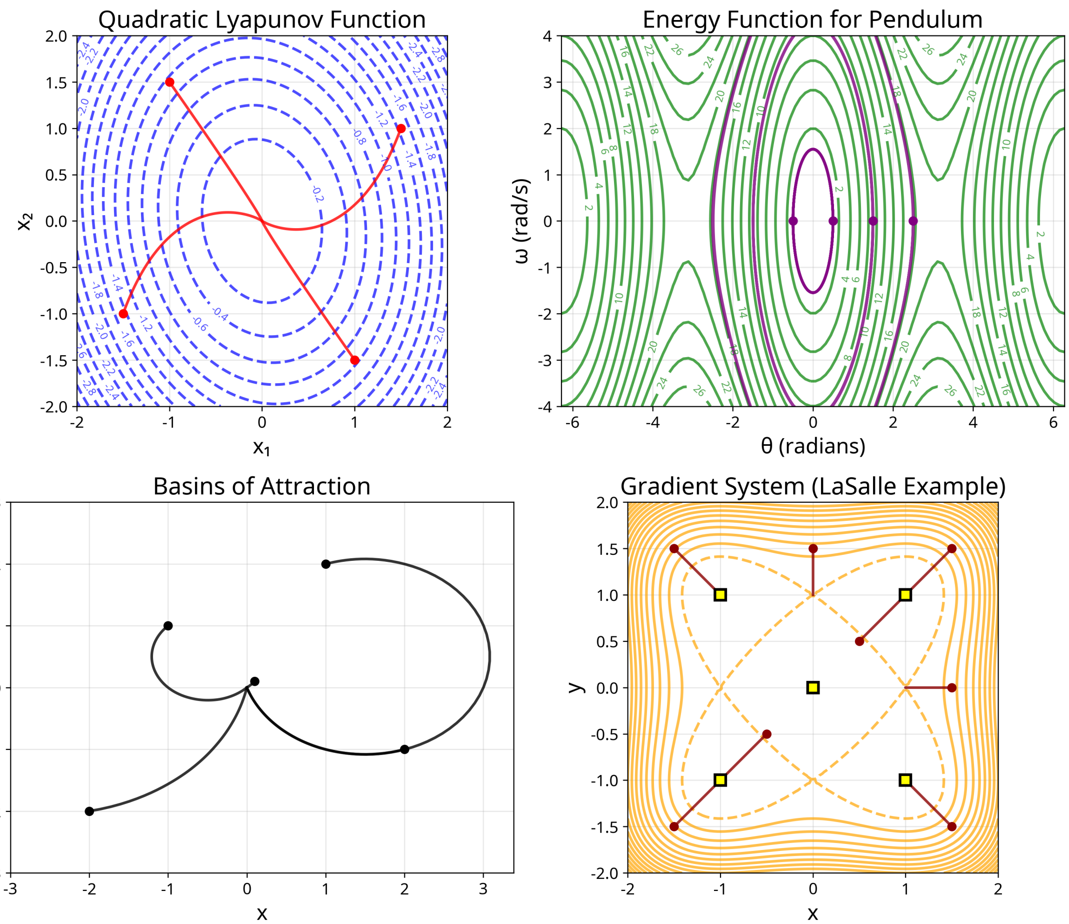

Figure 1: Lyapunov function examples: quadratic functions for linear systems, energy functions for conservative systems, basin of attraction analysis, and gradient system dynamics illustrating LaSalle invariance principle.

1.0.3 Lyapunov's Direct Method

Lyapunov's direct method (also called the second method) provides a way to determine stability without solving the differential equation explicitly. The method is based on constructing auxiliary functions, called Lyapunov functions, that capture the essential stability properties of the system.

1.0.4 Lyapunov Functions for Autonomous Systems

Consider the autonomous system $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ with an equilibrium at $\mathbf{x}^*$ (so $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$). A Lyapunov function is a scalar function $V(\mathbf{x})$ that satisfies certain properties related to the system's energy or distance from equilibrium.

Definition 1.1. A function $V : D \to \mathbb{R}$ is a Lyapunov function for the system $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ on domain D containing equilibrium $\mathbf{x}^*$ if:

1. $V(\mathbf{x}^*) = 0$
2. $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in D \setminus \{\mathbf{x}^*\}$ (positive definite)

3. V is continuously differentiable on D

The function $\dot{V}(\mathbf{x}) = \nabla V \cdot \mathbf{f}(\mathbf{x})$ is called the orbital derivative of V along system trajectories.

The orbital derivative measures how V changes along solution trajectories. If $\mathbf{x}(t)$ is a solution, then:

$$\frac{d}{dt}V(\mathbf{x}(t)) = \nabla V(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt} = \nabla V(\mathbf{x}(t)) \cdot \mathbf{f}(\mathbf{x}(t)) = \dot{V}(\mathbf{x}(t)) \tag{1}$$

Theorem 1.2. *Let $V(\mathbf{x})$ be a Lyapunov function for system $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ on domain D containing equilibrium $\mathbf{x}^*$. Then:*

1. *If $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in D$, then $\mathbf{x}^*$ is Lyapunov stable.*
2. *If $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in D \setminus \{\mathbf{x}^*\}$, then $\mathbf{x}^*$ is asymptotically stable.*
3. *If additionally $V(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$, then $\mathbf{x}^*$ is globally asymptotically stable.*

The intuition behind this theorem is that V acts like an energy function. If V decreases along trajectories ($\dot{V} < 0$), then solutions lose "energy" and must approach the minimum at $\mathbf{x}^*$. If V merely doesn't increase ($\dot{V} \leq 0$), solutions remain bounded but may not converge.

Example. Consider the linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ has eigenvalues with negative real parts. We can construct a quadratic Lyapunov function:

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x} \tag{2}$$

where $\mathbf{P}$ is a positive definite matrix satisfying the Lyapunov equation:

$$\mathbf{A}^T \mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{3}$$

for some positive definite matrix $\mathbf{Q}$.
The orbital derivative is:

$$\dot{V}(\mathbf{x}) = 2\mathbf{x}^T \mathbf{P}\mathbf{A}\mathbf{x} = -\mathbf{x}^T \mathbf{Q}\mathbf{x} < 0 \tag{4}$$

This proves global asymptotic stability of the origin.

1.0.5 Construction of Lyapunov Functions

Finding appropriate Lyapunov functions is often the most challenging aspect of stability analysis. Several systematic approaches exist:

Physical Energy: For mechanical systems, total energy (kinetic plus potential) often serves as a natural Lyapunov function. For electrical circuits, energy stored in capacitors and inductors provides similar functions.

Quadratic Forms: For systems near equilibria, quadratic functions $V(\mathbf{x}) = \mathbf{x}^T \mathbf{P}\mathbf{x}$ are often effective. The matrix $\mathbf{P}$ can be determined by solving Lyapunov equations or using optimization methods.

Sum of Squares: For polynomial systems, Lyapunov functions can be constructed as sums of squares of polynomials. This approach connects to semidefinite programming and computational methods.

Control Lyapunov Functions: In control theory, Lyapunov functions are designed to guide the construction of stabilizing feedback controllers.

1.0.6 LaSalle's Invariance Principle

While Lyapunov's direct method requires $\dot{V} < 0$ for asymptotic stability, many systems have Lyapunov functions where $\dot{V} \leq 0$ with equality on some set. LaSalle's invariance principle extends Lyapunov theory to handle these cases.

Theorem 1.3. *Let Ω be a compact positively invariant set for system $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$. Let $V : \Omega \to \mathbb{R}$ be continuously differentiable with $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega$.*

Define $E = \{\mathbf{x} \in \Omega : \dot{V}(\mathbf{x}) = 0\}$ and let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

This principle is particularly powerful for analyzing systems where energy is conserved along some directions but dissipated along others.

Example. Consider a damped pendulum:

$$\frac{d\theta}{dt} = \omega \tag{5}$$

$$\frac{d\omega}{dt} = -\sin\theta - c\omega \tag{6}$$

where $c > 0$ is the damping coefficient.

The total energy is:

$$V(\theta, \omega) = \frac{1}{2}\omega^2 + (1 - \cos\theta) \tag{7}$$

The orbital derivative is:

$$\dot{V} = \omega(-\sin\theta - c\omega) + \sin\theta \cdot \omega = -c\omega^2 \leq 0 \tag{8}$$

We have $\dot{V} = 0$ only when $\omega = 0$. On this set, $\frac{d\omega}{dt} = -\sin\theta$, which equals zero only at $\theta = 0, \pi, 2\pi, \ldots$

The largest invariant set in $\{\omega = 0\}$ consists of the equilibria $(\theta, \omega) = (2\pi k, 0)$ for integer k. By LaSalle's principle, all trajectories approach one of these equilibria.

Further analysis using linearization shows that $(0, 0)$ is stable while $(\pi, 0)$ is unstable, so trajectories approach $(0, 0)$ from a neighborhood and $(\pm 2\pi, 0)$ from trajectories that cross the separatrices.

1.0.7 Instability and Chetaev's Theorem

While Lyapunov theory provides tools for proving stability, proving instability requires different approaches. Chetaev's theorem offers a method for establishing instability using auxiliary functions.

Theorem 1.4. *Consider system $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ with equilibrium at origin. Suppose there exists a function $V(\mathbf{x})$ and a region U containing the origin such that:*

1. *$V(\mathbf{0}) = 0$*
2. *In U, the set $\{\mathbf{x} : V(\mathbf{x}) > 0\}$ is nonempty and $\dot{V}(\mathbf{x}) > 0$ whenever $V(\mathbf{x}) > 0$*
3. *Every neighborhood of the origin contains points where $V(\mathbf{x}) > 0$*

Then the origin is unstable.

The idea is to find a function that increases along some trajectories starting arbitrarily close to the equilibrium, forcing these trajectories to move away from equilibrium.

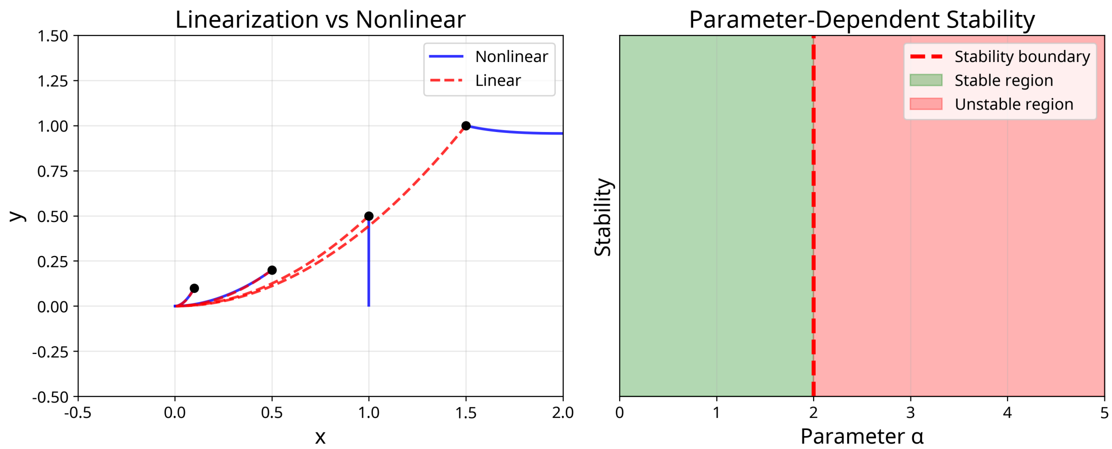

Figure 2: Stability analysis: (left) comparison of linearization with nonlinear behavior showing validity regions, (right) parameter-dependent stability boundaries demonstrating critical parameter values.

1.0.8 Basin of Attraction and Region of Stability

For asymptotically stable equilibria, the basin of attraction (or region of attraction) is the set of all initial conditions whose trajectories converge to the equilibrium. Determining this region is crucial for understanding the practical stability of systems.

1.0.9 Estimating Basins of Attraction

Lyapunov functions provide a systematic way to estimate basins of attraction. If $V(\mathbf{x})$ is a Lyapunov function with $\dot{V}(\mathbf{x}) < 0$ for $\mathbf{x} \neq \mathbf{x}^*$, then any level set $\{\mathbf{x} : V(\mathbf{x}) \leq c\}$ that doesn't contain other equilibria lies within the basin of attraction.

The largest such level set provides an estimate of the basin. While this estimate may be conservative, it gives a guaranteed region of stability.

Example. Consider the system:

$$\frac{dx}{dt} = -x + xy \tag{9}$$

$$\frac{dy}{dt} = -y - x^2 \tag{10}$$

The origin is an equilibrium. The linearization has matrix:

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

This shows local asymptotic stability. To estimate the basin of attraction, try the quadratic Lyapunov function:

$$V(x, y) = x^2 + y^2 \tag{12}$$

The orbital derivative is:

$$\dot{V} = 2x(-x + xy) + 2y(-y - x^2) = -2x^2 + 2x^2y - 2y^2 - 2x^2y = -2x^2 - 2y^2 < 0 \tag{13}$$

for $(x, y) \neq (0, 0)$. Since $V(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$, the origin is globally asymptotically stable.

1.0.10 Multiple Equilibria and Competing Basins

When systems have multiple stable equilibria, their basins of attraction partition the phase space. The boundaries between basins often contain unstable equilibria or limit cycles and represent separatrices in the dynamics.

Understanding these boundaries is crucial for predicting system behavior. Small perturbations that move initial conditions across basin boundaries can lead to dramatically different long-term behavior.

1.0.11 Converse Lyapunov Theorems

While Lyapunov's direct method provides sufficient conditions for stability, converse theorems establish that these conditions are also necessary. These results guarantee that stable systems always have Lyapunov functions, even if finding them explicitly may be difficult.

Theorem 1.5. *If the origin is asymptotically stable for system* $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$*, then there exists a Lyapunov function* $V(\mathbf{x})$ *such that* $V(\mathbf{x}) > 0$ *for* $\mathbf{x} \neq \mathbf{0}$ *and* $\dot{V}(\mathbf{x}) < 0$ *for* $\mathbf{x} \neq \mathbf{0}$ *in some neighborhood of the origin.*

Furthermore, if the origin is exponentially stable, then there exists a Lyapunov function satisfying:

$$\alpha_1 |\mathbf{x}|^2 \leq V(\mathbf{x}) \leq \alpha_2 |\mathbf{x}|^2 \tag{14}$$

$$\dot{V}(\mathbf{x}) \leq -\alpha_3 |\mathbf{x}|^2 \tag{15}$$

for positive constants $\alpha_1, \alpha_2, \alpha_3$.

These converse theorems provide theoretical completeness to Lyapunov theory and justify the search for Lyapunov functions in stability analysis.

1.0.12 Stability of Periodic Orbits

Extending stability analysis to periodic orbits requires modifications of the basic Lyapunov approach. The key insight is to study the behavior of nearby trajectories relative to the periodic orbit.

1.0.13 Poincaré Maps and Floquet Theory

For a periodic orbit $\mathbf{x}_p(t)$ with period T, we can analyze stability using a Poincaré map. Choose a cross-section Σ transverse to the orbit and define the map $P : \Sigma \to \Sigma$ that takes points to their next intersection with Σ.

The periodic orbit corresponds to a fixed point of P, and its stability is determined by the eigenvalues of DP (the multipliers). The orbit is stable if all multipliers have magnitude less than one.

Alternatively, Floquet theory analyzes the linearization around the periodic orbit. The fundamental matrix solution $\Phi(t)$ satisfies $\Phi(t + T) = \Phi(t)M$ where M is the monodromy matrix. The eigenvalues of M (Floquet multipliers) determine stability.

1.0.14 Lyapunov Functions for Periodic Orbits

Constructing Lyapunov functions for periodic orbits is more complex than for equilibria. One approach uses the distance to the orbit:

$$V(\mathbf{x}) = \min_{s \in [0,T]} |\mathbf{x} - \mathbf{x}_p(s)|^2 \tag{16}$$

However, this function may not be differentiable everywhere. Alternative approaches include using energy-like functions or constructing functions in orbital coordinates.

1.0.15 Input-to-State Stability

Modern control theory extends classical stability concepts to systems with inputs or disturbances. Input-to-state stability (ISS) provides a framework for analyzing how external inputs affect system stability.

Definition 1.6. System $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ is input-to-state stable if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all initial conditions $\mathbf{x}_0$ and inputs $\mathbf{u}(t)$:

$$|\mathbf{x}(t)| \leq \beta(|\mathbf{x}_0|, t) + \gamma(\sup_{0 \leq s \leq t} |\mathbf{u}(s)|) \tag{17}$$

Here $\mathcal{K}$ denotes class $\mathcal{K}$ functions (continuous, strictly increasing, with $\gamma(0) = 0$) and $\mathcal{KL}$ denotes class $\mathcal{KL}$ functions (decreasing in the second argument for each fixed first argument).

ISS captures the intuitive notion that bounded inputs should produce bounded outputs, with the bound depending continuously on the input magnitude.

1.0.16 Computational Methods in Stability Analysis

Modern computational tools have revolutionized stability analysis, enabling the study of high-dimensional systems and the construction of Lyapunov functions for complex nonlinear systems.

1.0.17 Sum of Squares Programming

For polynomial systems, Lyapunov functions can be constructed as sums of squares (SOS) of polynomials. This approach reformulates the search for Lyapunov functions as a semidefinite programming problem, which can be solved efficiently using interior-point methods.

The key insight is that a polynomial $p(\mathbf{x})$ is positive if and only if it can be written as:

$$p(\mathbf{x}) = \sum_{i=1}^{m} q_i(\mathbf{x})^2 \tag{18}$$

for some polynomials $q_i(\mathbf{x})$. This condition can be expressed as a semidefinite constraint on the coefficients.

1.0.18 Numerical Construction of Lyapunov Functions

For general nonlinear systems, numerical methods can construct piecewise-linear or radial basis function Lyapunov functions. These approaches discretize the state space and solve optimization problems to find functions satisfying the Lyapunov conditions.

Machine learning techniques, including neural networks, have also been applied to learn Lyapunov functions from simulation data. These methods show promise for high-dimensional systems where traditional approaches become computationally intractable.

Computational Note: The file `lecture6.py` implements various stability analysis methods, including Lyapunov function construction for linear systems, numerical basin of attraction estimation, and SOS-based methods for polynomial systems. The code demonstrates both theoretical concepts and practical computational techniques.

1.0.19 Applications in Control and Engineering

Stability theory forms the foundation of modern control system design. Controllers are designed not just to achieve desired performance but to guarantee stability under uncertainties and disturbances.

1.0.20 Lyapunov-Based Control Design

Control Lyapunov functions (CLFs) provide a systematic approach to controller synthesis. Given a system $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$, a CLF is a function $V(\mathbf{x})$ such that for each $\mathbf{x} \neq \mathbf{0}$, there exists a control $\mathbf{u}$ making $\dot{V} < 0$.

The control law can then be chosen to minimize $\dot{V}$, ensuring stability while optimizing performance criteria.

1.0.21 Robust Stability Analysis

Real systems always contain uncertainties in parameters, unmodeled dynamics, and external disturbances. Robust stability analysis extends Lyapunov methods to guarantee stability despite these uncertainties.

Techniques include: - **Quadratic Stability:** Using a single quadratic Lyapunov function for all possible parameter values - **Parameter-Dependent Lyapunov Functions:** Allowing the Lyapunov function to depend on uncertain parameters - **Integral Quadratic Constraints:** Incorporating information about the structure of uncertainties

This lecture has developed the fundamental theory and methods of stability analysis for dynamical systems. The key contributions include:

Lyapunov's Direct Method: Provides a systematic framework for analyzing stability without solving differential equations explicitly. The method's power lies in its generality and its ability to provide global stability results.

LaSalle's Invariance Principle: Extends Lyapunov theory to systems where energy is conserved along some directions. This principle is particularly valuable for mechanical and physical systems with natural conservation laws.

Basin of Attraction Analysis: Determines the region of initial conditions leading to stable behavior. Understanding these regions is crucial for predicting system behavior and designing robust controllers.

Computational Methods: Modern optimization and machine learning techniques enable stability analysis of complex, high-dimensional systems that were previously intractable.

Stability theory provides both theoretical insights and practical tools for system analysis and design. The methods developed here form the foundation for advanced topics in control theory, including adaptive control, robust control, and nonlinear control design.

The concepts introduced in this lecture will be essential for understanding the numerical methods and applications discussed in subsequent lectures. The interplay between stability theory and computational methods continues to drive advances in our ability to analyze and control complex dynamical systems.