Ordinary Differential Equations

Lecture Notes

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1 Lecture 5: Nonlinear Dynamics and Phase Plane Analysis

1.0.1 Introduction to Nonlinear Systems

Nonlinear differential equations represent the vast majority of mathematical models encountered in real-world applications. Unlike linear systems, which admit superposition and have well-understood solution structures, nonlinear systems exhibit a rich variety of behaviors that can include multiple equilibria, limit cycles, chaos, and sensitive dependence on initial conditions. The study of nonlinear dynamics has revolutionized our understanding of complex systems across disciplines ranging from physics and biology to economics and engineering.

The general autonomous nonlinear system in the plane takes the form:

$$\frac{dx}{dt} = f(x, y) \tag{1}$$

$$\frac{dy}{dt} = g(x, y) \tag{2}$$

where f and g are nonlinear functions of the state variables. The absence of explicit time dependence in autonomous systems allows us to focus on the geometric structure of the phase space and the qualitative behavior of trajectories.

The fundamental challenge in nonlinear dynamics is that exact analytical solutions are rarely available. Instead, we rely on qualitative methods that reveal the essential features of system behavior without requiring explicit solution formulas. These methods include phase plane analysis, linearization near equilibria, energy methods, and geometric approaches that exploit the structure of the vector field.

1.0.2 Fundamental Differences from Linear Systems

Nonlinear systems exhibit phenomena that are impossible in linear systems. The principle of superposition fails, meaning that linear combinations of solutions are generally not solutions. This breakdown of linearity leads to several distinctive features:

Multiple Equilibria: While linear systems have at most one equilibrium point (excluding degenerate cases), nonlinear systems can have arbitrarily many equilibria. Each equilibrium can have different stability properties, creating a complex landscape of attracting and repelling regions in phase space.

Limit Cycles: Nonlinear systems can exhibit isolated periodic orbits called limit cycles. These are closed trajectories in phase space that attract or repel nearby trajectories. Limit cycles represent

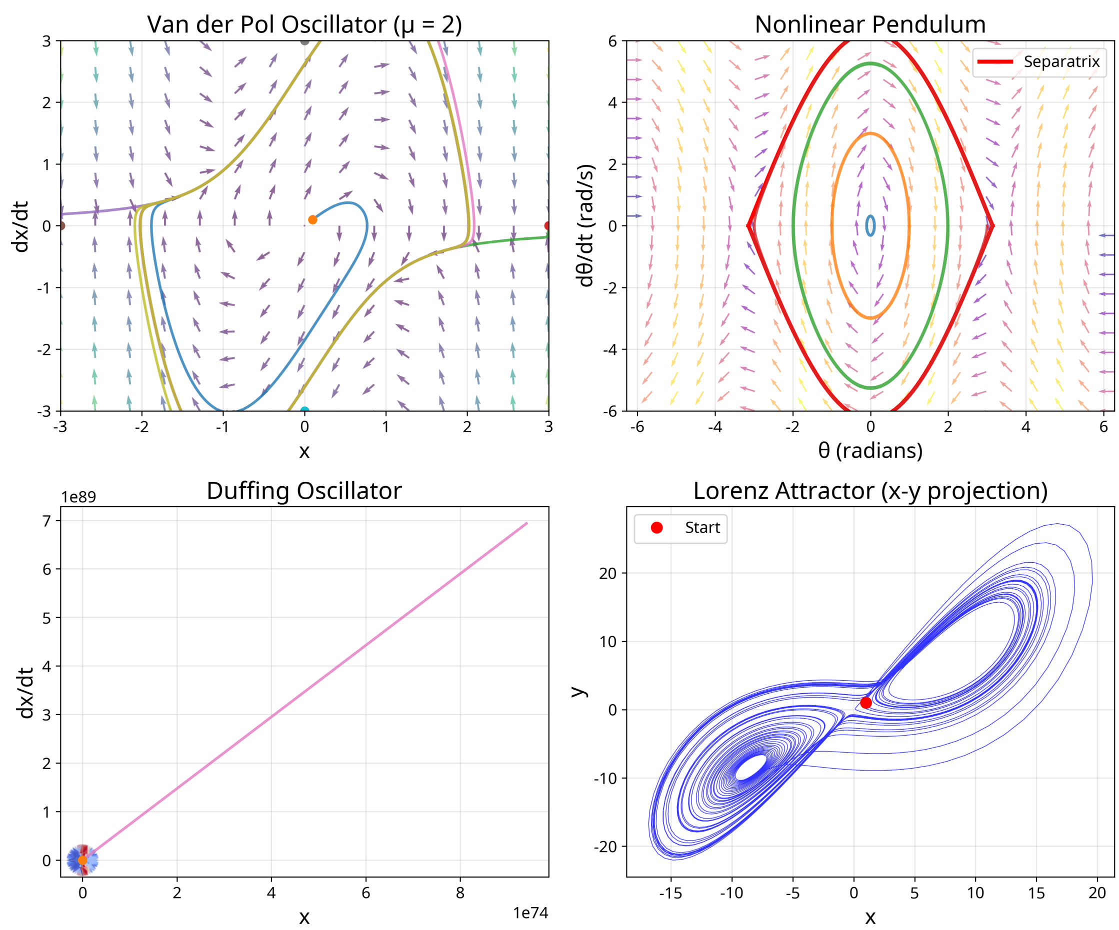

Figure 1: Nonlinear system examples: Van der Pol oscillator limit cycle, nonlinear pendulum with separatrices, Duffing oscillator multiple equilibria, and Lorenz attractor chaotic dynamics.

sustained oscillations that are structurally stable, meaning they persist under small perturbations of the system parameters.

Separatrices: Special trajectories called separatrices divide phase space into regions with qualitatively different behavior. These curves often connect saddle points and form boundaries between basins of attraction for different equilibria or limit cycles.

Sensitive Dependence: Some nonlinear systems exhibit chaotic behavior, where trajectories starting from nearby initial conditions diverge exponentially over time. This sensitive dependence on initial conditions makes long-term prediction impossible despite the deterministic nature of the equations.

1.0.3 Phase Plane Analysis

The phase plane provides the primary tool for analyzing two-dimensional nonlinear systems. By plotting trajectories in the (x, y) plane, we can visualize the global behavior of the system and identify key features such as equilibria, limit cycles, and separatrices.

1.0.4 Nullclines and Flow Patterns

Nullclines play a crucial role in organizing the phase plane structure. The x-nullclines are curves where $\frac{dx}{dt} = 0$, so $f(x, y) = 0$. Similarly, y-nullclines satisfy $g(x, y) = 0$. These curves divide the phase plane into regions where the flow has consistent direction.

On x-nullclines, trajectories move purely vertically since $\frac{dx}{dt} = 0$ but $\frac{dy}{dt} \neq 0$ in general. On y-nullclines, motion is purely horizontal. The intersections of x- and y-nullclines correspond to equilibrium points where both derivatives vanish.

The direction of flow in each region can be determined by evaluating the signs of $f(x, y)$ and $g(x, y)$. This creates a systematic method for sketching the global flow pattern without solving the differential equation explicitly.

Example Van der Pol Oscillator. The Van der Pol oscillator is a classic example of a nonlinear system with a limit cycle:

$$\frac{dx}{dt} = y \tag{3}$$

$$\frac{dy}{dt} = \mu(1 - x^2)y - x \tag{4}$$

The x-nullcline is $y = 0$, and the y-nullcline is the cubic curve $y = \frac{x}{\mu(1-x^2)}$ for $x \neq \pm 1$.

For $\mu > 0$, the origin is an unstable focus, and the system has a unique, stable limit cycle. The parameter μ controls the nonlinearity strength: for small μ, the limit cycle is nearly circular, while for large μ, it becomes increasingly distorted with fast and slow phases.

1.0.5 Poincaré-Bendixson Theory

The Poincaré-Bendixson theorem provides fundamental results about the long-term behavior of trajectories in two-dimensional systems. This theorem is unique to planar systems and does not extend to higher dimensions.

Theorem 1.1 (Poincaré-Bendixson Theorem)**.** *Let R be a bounded region in the plane with the property that trajectories cannot escape from R. If R contains no equilibrium points, then every trajectory in R approaches a periodic orbit as $t \to \infty$.*

More generally, if a trajectory is trapped in a bounded region R and does not approach an equilibrium point, then its ω-limit set is either:

1. *A periodic orbit, or*

2. *A graphic (a union of equilibria and trajectories connecting them)*

This theorem has profound implications for planar dynamics. It guarantees that bounded trajectories that don't approach equilibria must exhibit periodic behavior. This rules out chaotic behavior in two-dimensional autonomous systems, as chaos requires at least three dimensions.

The theorem also provides a systematic method for proving the existence of limit cycles. By constructing appropriate trapping regions and showing they contain no equilibria, we can guarantee the existence of periodic orbits without finding them explicitly.

1.0.6 Equilibria and Linear Stability Analysis

Equilibrium points are solutions where the vector field vanishes: $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$. The behavior near equilibria is crucial for understanding global dynamics, as equilibria often organize the phase space structure.

1.0.7 Linearization and the Jacobian Matrix

Near an equilibrium point (x^*, y^*), we can approximate the nonlinear system by its linearization. Let $u = x - x^*$ and $v = y - y^*$ represent small displacements from equilibrium. Taylor expansion gives:

$$\frac{du}{dt} = f_x(x^*, y^*)u + f_y(x^*, y^*)v + O(u^2, v^2, uv) \tag{5}$$

$$\frac{dv}{dt} = g_x(x^*, y^*)u + g_y(x^*, y^*)v + O(u^2, v^2, uv) \tag{6}$$

The linear approximation is:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{J}(x^*, y^*)\begin{pmatrix} u \\ v \end{pmatrix} \tag{7}$$

where the Jacobian matrix is:

$$\mathbf{J}(x^*, y^*) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\Bigg|_{(x^*, y^*)} \tag{8}$$

1.0.8 Classification of Equilibria

The eigenvalues of the Jacobian matrix determine the local behavior near equilibria. Let λ_1 and λ_2 be the eigenvalues, with $\tau = \lambda_1 + \lambda_2$ (trace) and $\Delta = \lambda_1 \lambda_2$ (determinant).

Hyperbolic Equilibria: When both eigenvalues have nonzero real parts, the equilibrium is hyperbolic. The classification depends on the signs of the eigenvalues:

- **Stable Node:** $\lambda_1, \lambda_2 < 0$ (both real and negative)
- **Unstable Node:** $\lambda_1, \lambda_2 > 0$ (both real and positive)
- **Saddle Point:** $\lambda_1 < 0 < \lambda_2$ (real with opposite signs)
- **Stable Focus:** $\mathrm{Re}(\lambda_{1,2}) < 0$ (complex with negative real part)
- **Unstable Focus:** $\mathrm{Re}(\lambda_{1,2}) > 0$ (complex with positive real part)

Non-hyperbolic Equilibria: When one or both eigenvalues have zero real part, linearization fails to determine stability. These cases require nonlinear analysis:

- **Center:** $\lambda_{1,2} = \pm i\omega$ (purely imaginary)
- **Degenerate Cases:** One or both eigenvalues equal zero

Theorem 1.2. *Near a hyperbolic equilibrium point, the nonlinear system is topologically equivalent to its linearization. This means there exists a homeomorphism that maps trajectories of the nonlinear system to trajectories of the linear system, preserving the direction of time.*

This theorem justifies the use of linear stability analysis for hyperbolic equilibria. The local phase portrait of the nonlinear system near a hyperbolic equilibrium has the same qualitative structure as the linearized system.

1.0.9 Bifurcation Theory

Bifurcations occur when small changes in system parameters cause qualitative changes in the dynamics. At bifurcation points, the system undergoes structural changes such as the creation or destruction of equilibria, changes in stability, or the birth of periodic orbits.

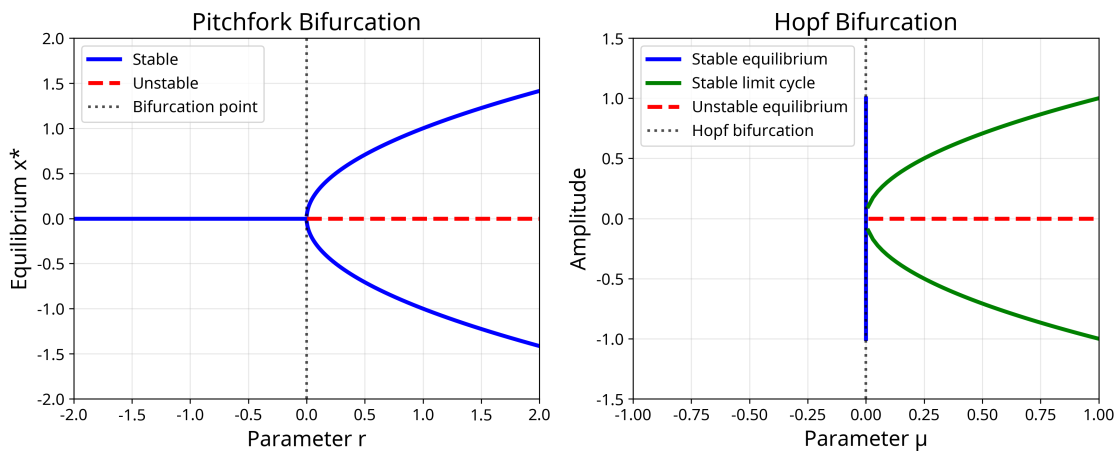

Figure 2: Bifurcation diagrams: (left) pitchfork bifurcation showing symmetry breaking, (right) Hopf bifurcation demonstrating transition from equilibrium to oscillatory behavior.

1.0.10 Local Bifurcations

Local bifurcations occur when equilibria change stability or when new equilibria are created or destroyed. The most common local bifurcations in planar systems are:

Saddle-Node Bifurcation: Two equilibria (one stable, one unstable) collide and annihilate each other. This is the generic mechanism for the creation and destruction of equilibria.

Consider the normal form:

$$\frac{dx}{dt} = \mu - x^2 \tag{9}$$

For $\mu > 0$, there are two equilibria at $x = \pm\sqrt{\mu}$. At $\mu = 0$, they collide at the origin. For $\mu < 0$, no equilibria exist.

Transcritical Bifurcation: Two equilibria exchange stability as they pass through each other. This bifurcation preserves the number of equilibria but changes their stability properties.

The normal form is:

$$\frac{dx}{dt} = \mu x - x^2 \tag{10}$$

There are always two equilibria at $x = 0$ and $x = \mu$. Their stability exchanges at $\mu = 0$.

Pitchfork Bifurcation: A single equilibrium splits into three equilibria. This bifurcation often occurs in systems with symmetry.

The supercritical pitchfork has normal form:

$$\frac{dx}{dt} = \mu x - x^3 \tag{11}$$

For $\mu < 0$, there is one stable equilibrium at $x = 0$. For $\mu > 0$, the origin becomes unstable and two new stable equilibria appear at $x = \pm\sqrt{\mu}$.

1.0.11 Hopf Bifurcation

The Hopf bifurcation is particularly important as it represents the transition between equilibrium and oscillatory behavior. It occurs when a pair of complex conjugate eigenvalues crosses the imaginary axis.

Theorem 1.3. *Consider a system* $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mu)$ *with an equilibrium at* $\mathbf{x} = \mathbf{0}$ *for all* μ. *Suppose the Jacobian* $\mathbf{J}(\mu)$ *has eigenvalues* $\lambda(\mu) = \alpha(\mu) \pm i\beta(\mu)$ *with:*

1. $\alpha(0) = 0$, $\beta(0) = \omega_0 \neq 0$

2. $\left.\frac{d\alpha}{d\mu}\right|_{\mu=0} \neq 0$

Then a unique branch of periodic orbits bifurcates from the equilibrium at $\mu = 0$. *The bifurcation is supercritical (stable limit cycle) if the first Lyapunov coefficient is negative, and subcritical (unstable limit cycle) if it is positive.*

The Hopf bifurcation explains the emergence of oscillations in many physical systems. Examples include the onset of oscillations in chemical reactions, predator-prey cycles in ecology, and business cycles in economics.

1.0.12 Conservative Systems and Hamiltonian Dynamics

Conservative systems form an important class of nonlinear systems where energy is preserved. These systems arise naturally in mechanics and have special geometric properties that constrain their dynamics.

1.0.13 Hamiltonian Systems

A Hamiltonian system in the plane has the form:

$$\frac{dx}{dt} = \frac{\partial H}{\partial y} \tag{12}$$

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x} \tag{13}$$

where $H(x, y)$ is the Hamiltonian function, typically representing total energy.

The key property of Hamiltonian systems is energy conservation:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} = 0 \tag{14}$$

This means trajectories lie on level curves of the Hamiltonian, $H(x, y) = \text{constant}$.

Example. The equation for a nonlinear pendulum is:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \tag{15}$$

Converting to first-order form with $x = \theta$ and $y = \frac{d\theta}{dt}$:

$$\frac{dx}{dt} = y \tag{16}$$

$$\frac{dy}{dt} = -\frac{g}{l}\sin x \tag{17}$$

The Hamiltonian is:

$$H(x, y) = \frac{1}{2}y^2 - \frac{g}{l}\cos x \tag{18}$$

Level curves of H give the phase portrait. Small oscillations correspond to closed orbits around the stable equilibrium at $(0, 0)$. Large energy trajectories include separatrices connecting saddle points at $(\pm\pi, 0)$.

1.0.14 Liouville's Theorem and Phase Space Volume

Hamiltonian systems preserve phase space volume, a property known as Liouville's theorem. This has profound implications for the dynamics:

Theorem 1.4. *The flow of a Hamiltonian system preserves phase space volume. If $D(t)$ is a region in phase space evolved under the Hamiltonian flow, then:*

$$\frac{d}{dt}\int_{D(t)} dx\, dy = 0 \tag{19}$$

This theorem implies that Hamiltonian systems cannot have attracting equilibria or limit cycles, as these would require phase space volume to contract. All equilibria in Hamiltonian systems are centers or saddles.

1.0.15 Gradient Systems and Lyapunov Functions

Gradient systems represent another special class where the vector field derives from a scalar potential function. These systems have the form:

$$\frac{d\mathbf{x}}{dt} = -\nabla V(\mathbf{x}) \tag{20}$$

where $V(\mathbf{x})$ is a potential function.

1.0.16 Properties of Gradient Systems

Gradient systems have several distinctive properties:

Monotonic Energy Decrease: The potential function V decreases monotonically along trajectories:

$$\frac{dV}{dt} = \nabla V \cdot \frac{d\mathbf{x}}{dt} = -|\nabla V|^2 \leq 0 \tag{21}$$

No Closed Orbits: Since V decreases along trajectories, closed orbits are impossible (except at equilibria where $\nabla V = 0$).

Simple Equilibria: All equilibria are either sinks or sources, determined by the Hessian matrix of V. Saddle points cannot occur in gradient systems.

Example. Consider the potential:

$$V(x, y) = x^4 + y^4 - 2x^2 - 2y^2 \tag{22}$$

The gradient system is:

$$\frac{dx}{dt} = -(4x^3 - 4x) = -4x(x^2 - 1) \tag{23}$$

$$\frac{dy}{dt} = -(4y^3 - 4y) = -4y(y^2 - 1) \tag{24}$$

Equilibria occur at $(\pm 1, \pm 1)$ and $(0, 0)$. The Hessian analysis shows that $(\pm 1, \pm 1)$ are stable nodes (local minima of V) while $(0, 0)$ is an unstable node (local maximum of V).

1.0.17 Computational Methods for Nonlinear Analysis

While analytical methods provide fundamental insights, computational tools are essential for studying complex nonlinear systems. Modern software packages enable detailed phase portrait analysis, bifurcation studies, and numerical continuation of solution branches.

1.0.18 Numerical Phase Portrait Construction

Constructing accurate phase portraits requires careful numerical integration of trajectories from multiple initial conditions. Key considerations include:

Initial Condition Selection: Strategic placement of initial conditions near equilibria, along nullclines, and in different regions of phase space ensures comprehensive coverage of the dynamics.

Integration Methods: Adaptive step-size methods like Runge-Kutta-Fehlberg provide good accuracy while maintaining computational efficiency. For Hamiltonian systems, symplectic integrators preserve energy conservation properties.

Long-term Integration: Some features like limit cycles or chaotic attractors require long integration times to become apparent. Careful monitoring of numerical accuracy is essential for reliable results.

Computational Note: The file `lecture5.py` contains comprehensive implementations for phase portrait analysis, including nullcline computation, equilibrium finding, linear stability analysis, and bifurcation detection. The code demonstrates both analytical calculations and numerical methods for studying nonlinear dynamics.

1.0.19 Applications in Science and Engineering

Nonlinear dynamics appears throughout science and engineering, providing models for phenomena ranging from mechanical vibrations to biological rhythms. Understanding nonlinear behavior is crucial for predicting and controlling complex systems.

1.0.20 Mechanical Systems

Nonlinear mechanical systems exhibit rich dynamics including multiple equilibria, limit cycles, and chaos. The Duffing oscillator, with equation:

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma\cos(\omega t) \tag{25}$$

demonstrates phenomena such as jump resonance, hysteresis, and chaotic motion depending on parameter values.

1.0.21 Biological Systems

Population dynamics, neural networks, and biochemical reactions all exhibit nonlinear behavior. The FitzHugh-Nagumo model for neural excitation:

$$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I \tag{26}$$

$$\frac{dw}{dt} = \epsilon(v + a - bw) \tag{27}$$

captures the essential features of action potential generation and propagation in neurons.

1.0.22 Chemical Reactions

Autocatalytic chemical reactions can exhibit oscillations, bistability, and spatial patterns. The Brusselator model:

$$\frac{dx}{dt} = A - (B + 1)x + x^2y \tag{28}$$

$$\frac{dy}{dt} = Bx - x^2y \tag{29}$$

demonstrates how simple reaction schemes can produce complex temporal dynamics.

This lecture has introduced the fundamental concepts and methods of nonlinear dynamics. Key insights include:

Qualitative Methods: Phase plane analysis, nullcline construction, and linearization provide powerful tools for understanding nonlinear systems without requiring explicit solutions.

Bifurcation Theory: Parameter-dependent changes in system structure reveal how complex behavior emerges from simple models. Bifurcations organize the parameter space and predict transitions between different dynamical regimes.

Special System Classes: Conservative and gradient systems have distinctive properties that constrain their possible behaviors. Understanding these constraints helps classify and analyze specific systems.

Computational Integration: Numerical methods extend analytical insights to complex systems that resist exact analysis. Modern computational tools enable detailed exploration of parameter space and long-term dynamics.

The study of nonlinear dynamics reveals that deterministic systems can exhibit extraordinarily complex behavior. This complexity is not due to external randomness but emerges from the intrinsic nonlinear interactions within the system. Understanding these mechanisms provides insight into phenomena across all areas of science and engineering, from the onset of turbulence in fluid flow to the dynamics of ecosystems and financial markets.

The next lecture will build on these foundations by examining stability theory and Lyapunov methods, which provide rigorous tools for analyzing the long-term behavior of nonlinear systems.