Ordinary Differential Equations

Lecture Notes

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1 Lecture 4: Eigenvalue Methods and Diagonalization

1.0.1 Introduction to Eigenvalue Methods

Eigenvalue methods provide the most powerful and systematic approach to solving linear systems of differential equations. These methods not only yield explicit solutions but also reveal the fundamental structure underlying the system's behavior. The eigenvalue-eigenvector decomposition transforms complex coupled systems into collections of independent, simpler equations.

The central idea is to find special directions in the state space—the eigenvector directions—along which the system evolves in the simplest possible way. Along these directions, the system behaves like a one-dimensional equation with exponential solutions.

For the linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, we seek solutions of the form:

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} \tag{1}$$

where $\mathbf{v}$ is a constant vector and λ is a constant scalar. Substituting into the differential equation:

$$\lambda \mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t} \tag{2}$$

This leads to the eigenvalue problem:

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \tag{3}$$

The scalars λ are eigenvalues and the corresponding vectors $\mathbf{v}$ are eigenvectors of matrix $\mathbf{A}$.

1.0.2 The Eigenvalue Problem

4.1: Eigenvalues and Eigenvectors For an $n \times n$ matrix $\mathbf{A}$, a scalar λ is an eigenvalue and a nonzero vector $\mathbf{v}$ is a corresponding eigenvector if:

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \tag{4}$$

Equivalently, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, which has nontrivial solutions if and only if:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{5}$$

This equation is called the characteristic equation, and its left side is the characteristic polynomial.

1.0.3 Computing Eigenvalues

The characteristic polynomial of an $n \times n$ matrix is a polynomial of degree n:

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 \tag{6}$$

The coefficients are related to the matrix elements through: - $c_{n-1} = (-1)^{n-1}\text{tr}(\mathbf{A})$ (trace) - $c_0 = \det(\mathbf{A})$ (determinant)

For a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - \text{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) \tag{7}$$

The eigenvalues are:

$$\lambda_{1,2} = \frac{\text{tr}(\mathbf{A}) \pm \sqrt{(\text{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A})}}{2} \tag{8}$$

1.0.4 Computing Eigenvectors

Once eigenvalues are found, eigenvectors are computed by solving:

$$(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0} \tag{9}$$

This is a homogeneous linear system. The eigenvector is determined up to a scalar multiple, so we typically normalize it or choose a convenient scaling.

4.1: Complete Eigenvalue Analysis Consider the matrix:

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \tag{10}$$

Step 1: Find eigenvalues

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) - 2 = \lambda^2 - 5\lambda + 4 \tag{11}$$

Factoring: $(\lambda - 4)(\lambda - 1) = 0$, so $\lambda_1 = 4$, $\lambda_2 = 1$.
Step 2: Find eigenvectors
For $\lambda_1 = 4$:

$$(\mathbf{A} - 4\mathbf{I})\mathbf{v}_1 = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\mathbf{v}_1 = \mathbf{0} \tag{12}$$

This gives $-v_1 + v_2 = 0$, so $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
For $\lambda_2 = 1$:

$$(\mathbf{A} - \mathbf{I})\mathbf{v}_2 = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\mathbf{v}_2 = \mathbf{0} \tag{13}$$

This gives $2v_1 + v_2 = 0$, so $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

1.0.5 Diagonalization

When a matrix has a complete set of linearly independent eigenvectors, it can be diagonalized.

4.1: Diagonalization Theorem An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if it has n linearly independent eigenvectors. In this case:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{14}$$

where $\mathbf{P}$ is the matrix of eigenvectors and $\mathbf{D}$ is the diagonal matrix of eigenvalues:

$$\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{15}$$

1.0.6 Solution via Diagonalization

When $\mathbf{A}$ is diagonalizable, the solution to $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ can be written as:

$$\mathbf{x}(t) = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{x}_0 \tag{16}$$

Since $\mathbf{D}$ is diagonal:

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \tag{17}$$

This shows that the solution is a linear combination of exponential functions with rates determined by the eigenvalues.

1.0.7 Modal Coordinates

The transformation $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ introduces modal coordinates. In these coordinates, the system becomes:

$$\frac{d\mathbf{y}}{dt} = \mathbf{D}\mathbf{y} \tag{18}$$

This is a decoupled system:

$$\frac{dy_1}{dt} = \lambda_1 y_1 \tag{19}$$

$$\frac{dy_2}{dt} = \lambda_2 y_2 \tag{20}$$

$$\vdots \tag{21}$$

$$\frac{dy_n}{dt} = \lambda_n y_n \tag{22}$$

Each modal coordinate evolves independently according to $y_i(t) = y_i(0)e^{\lambda_i t}$.

1.0.8 Complex Eigenvalues

When the coefficient matrix is real but has complex eigenvalues, they occur in conjugate pairs. This leads to oscillatory solutions.

1.0.9 Complex Exponentials and Real Solutions

If $\lambda = \alpha + i\beta$ is a complex eigenvalue with eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$, then:

$$\mathbf{x}(t) = e^{(\alpha+i\beta)t}(\mathbf{u} + i\mathbf{w}) = e^{\alpha t}[(\mathbf{u} + i\mathbf{w})(\cos(\beta t) + i\sin(\beta t))] \tag{23}$$

Expanding and taking real and imaginary parts:

$$\mathbf{x}_1(t) = e^{\alpha t}[\mathbf{u}\cos(\beta t) - \mathbf{w}\sin(\beta t)] \tag{24}$$
$$\mathbf{x}_2(t) = e^{\alpha t}[\mathbf{w}\cos(\beta t) + \mathbf{u}\sin(\beta t)] \tag{25}$$

These are two linearly independent real solutions.

4.2: System with Complex Eigenvalues Consider:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{26}$$

The characteristic equation is:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 1 = 0 \tag{27}$$

Eigenvalues: $\lambda = \pm i$
For $\lambda = i$:

$$(\mathbf{A} - i\mathbf{I})\mathbf{v} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\mathbf{v} = \mathbf{0} \tag{28}$$

This gives $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
So $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
With $\alpha = 0$ and $\beta = 1$, the real solutions are:

$$\mathbf{x}_1(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \tag{29}$$
$$\mathbf{x}_2(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \tag{30}$$

General solution:

$$\mathbf{x}(t) = c_1\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \tag{31}$$

1.0.10 Repeated Eigenvalues

When an eigenvalue has algebraic multiplicity greater than its geometric multiplicity, the matrix is not diagonalizable. In this case, we need generalized eigenvectors.

1.0.11 Geometric vs. Algebraic Multiplicity

4.2: Multiplicities For an eigenvalue λ:

- **Algebraic multiplicity**: The multiplicity of λ as a root of the characteristic polynomial
- **Geometric multiplicity**: The dimension of the eigenspace, i.e., $\dim(\text{null}(\mathbf{A} - \lambda\mathbf{I}))$

The geometric multiplicity is always less than or equal to the algebraic multiplicity.

1.0.12 Generalized Eigenvectors

When the geometric multiplicity is less than the algebraic multiplicity, we find generalized eigenvectors.

4.3: Generalized Eigenvectors For an eigenvalue λ with algebraic multiplicity m, the generalized eigenvectors of rank k satisfy:

$$(\mathbf{A} - \lambda\mathbf{I})^k\mathbf{v} = \mathbf{0} \tag{32}$$

The ordinary eigenvectors are generalized eigenvectors of rank 1.

1.0.13 Jordan Canonical Form

When a matrix is not diagonalizable, it can be transformed to Jordan canonical form:

$$\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1} \tag{33}$$

where $\mathbf{J}$ is a block diagonal matrix with Jordan blocks:

$$\mathbf{J}_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \tag{34}$$

4.3: Repeated Eigenvalue Case Consider:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \tag{35}$$

The characteristic equation is:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (2 - \lambda)^2 = 0 \tag{36}$$

So $\lambda = 2$ with algebraic multiplicity 2.
For the eigenspace:

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0} \tag{37}$$

This gives only one linearly independent eigenvector: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For the generalized eigenvector, we solve:

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 \tag{38}$$

This gives $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The solutions are:

$$\mathbf{x}_1(t) = e^{2t}\mathbf{v}_1 = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{39}$$

$$\mathbf{x}_2(t) = e^{2t}(t\mathbf{v}_1 + \mathbf{v}_2) = e^{2t}\begin{pmatrix} t \\ 1 \end{pmatrix} \tag{40}$$

1.0.14 Applications to Mechanical Systems

Eigenvalue methods are particularly powerful for analyzing mechanical systems with multiple degrees of freedom.

1.0.15 Normal Modes of Vibration

Consider a system of n masses connected by springs. The equations of motion are:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{41}$$

where $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix.

Assuming solutions of the form $\mathbf{q}(t) = \mathbf{v}e^{i\omega t}$:

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{v} = \mathbf{0} \tag{42}$$

This is a generalized eigenvalue problem:

$$\mathbf{K}\mathbf{v} = \omega^2\mathbf{M}\mathbf{v} \tag{43}$$

The eigenvalues ω_i^2 are the squares of the natural frequencies, and the eigenvectors $\mathbf{v}_i$ are the mode shapes.

1.0.16 Modal Analysis

The general solution is a superposition of normal modes:

$$\mathbf{q}(t) = \sum_{i=1}^{n}(A_i\cos(\omega_i t) + B_i\sin(\omega_i t))\mathbf{v}_i \tag{44}$$

Each mode oscillates independently at its natural frequency. This decomposition is fundamental to understanding vibrations in mechanical systems.

Computational Note: The file `lecture4.py` contains comprehensive implementations of eigenvalue computation, diagonalization procedures, and modal analysis for mechanical systems.

1.0.17 Stability Analysis via Eigenvalues

The eigenvalues of the coefficient matrix completely determine the stability of linear systems.

4.2: Stability Criteria For the linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$:
Asymptotically Stable: All eigenvalues have negative real parts

$$\text{Re}(\lambda_i) < 0 \text{ for all } i \tag{45}$$

Marginally Stable: All eigenvalues have non-positive real parts, with simple eigenvalues on the imaginary axis

$$\text{Re}(\lambda_i) \leq 0 \text{ for all } i, \text{ with simple eigenvalues when } \text{Re}(\lambda_i) = 0 \tag{46}$$

Unstable: At least one eigenvalue has positive real part

$$\text{Re}(\lambda_i) > 0 \text{ for some } i \tag{47}$$

1.0.18 Routh-Hurwitz Criteria

For determining stability without explicitly computing eigenvalues, the Routh-Hurwitz criteria provide necessary and sufficient conditions based on the coefficients of the characteristic polynomial.

For a polynomial $p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ with $a_n > 0$, all roots have negative real parts if and only if all the Hurwitz determinants are positive:

$$H_1 = a_{n-1}, \quad H_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}, \quad H_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix}, \ldots \tag{48}$$

1.0.19 Numerical Methods for Eigenvalue Problems

Computing eigenvalues and eigenvectors numerically is a fundamental problem in computational linear algebra.

1.0.20 Power Method

The power method finds the dominant eigenvalue (largest in absolute value) and its corresponding eigenvector.

4.1: Power Method Given matrix $\mathbf{A}$ and initial vector $\mathbf{v}_0$:
1. For $k = 0, 1, 2, \ldots$:

$$\mathbf{w}_{k+1} = \mathbf{A}\mathbf{v}_k \tag{49}$$

$$\mathbf{v}_{k+1} = \frac{\mathbf{w}_{k+1}}{\|\mathbf{w}_{k+1}\|} \tag{50}$$

$$\lambda_{k+1} = \mathbf{v}_{k+1}^T\mathbf{A}\mathbf{v}_{k+1} \tag{51}$$

2. Continue until convergence

The sequence λ_k converges to the dominant eigenvalue, and $\mathbf{v}_k$ converges to the corresponding eigenvector.

1.0.21 QR Algorithm

The QR algorithm is the most widely used method for computing all eigenvalues of a matrix.

4.2: QR Algorithm Given matrix $\mathbf{A}_0 = \mathbf{A}$:
1. For $k = 0, 1, 2, \ldots$:

$$\mathbf{A}_k = \mathbf{Q}_k\mathbf{R}_k \quad \text{(QR decomposition)} \tag{52}$$

$$\mathbf{A}_{k+1} = \mathbf{R}_k\mathbf{Q}_k \tag{53}$$

2. Continue until $\mathbf{A}_k$ converges to upper triangular form
The diagonal elements of the limit matrix are the eigenvalues.

1.0.22 Computational Considerations

- **Conditioning**: Eigenvalue problems can be ill-conditioned when eigenvalues are close together - **Deflation**: After finding one eigenvalue, deflation techniques can be used to find others - **Specialized methods**: Symmetric matrices, sparse matrices, and other special structures have specialized algorithms

1.0.23 Advanced Topics

1.0.24 Matrix Functions

Beyond the matrix exponential, other matrix functions arise in applications:

Matrix Sine and Cosine:

$$\sin(\mathbf{A}t) = \sum_{k=0}^{\infty} \frac{(-1)^k(\mathbf{A}t)^{2k+1}}{(2k+1)!} \tag{54}$$

$$\cos(\mathbf{A}t) = \sum_{k=0}^{\infty} \frac{(-1)^k(\mathbf{A}t)^{2k}}{(2k)!} \tag{55}$$

Matrix Square Root: $\mathbf{A}^{1/2}$ such that $(\mathbf{A}^{1/2})^2 = \mathbf{A}$
Matrix Logarithm: $\log(\mathbf{A})$ such that $e^{\log(\mathbf{A})} = \mathbf{A}$

1.0.25 Pseudospectra

For non-normal matrices, eigenvalues can be highly sensitive to perturbations. Pseudospectra provide a more robust analysis tool:

$$\sigma_\epsilon(\mathbf{A}) = \{\lambda \in \mathbb{C} : \|(\lambda\mathbf{I} - \mathbf{A})^{-1}\| \geq 1/\epsilon\} \tag{56}$$

This set includes all points that are eigenvalues of some matrix within distance ϵ of $\mathbf{A}$.

This lecture has developed the complete eigenvalue theory for linear systems:

Eigenvalue Problem: The fundamental equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ reveals the natural directions and rates of evolution for linear systems. Computing eigenvalues and eigenvectors provides the key to understanding system behavior.

Diagonalization: When a matrix has a complete set of eigenvectors, it can be diagonalized, leading to decoupled modal equations. This transformation reveals the independent modes of the system.

Complex Eigenvalues: Complex eigenvalues lead to oscillatory solutions with exponential envelopes. The real and imaginary parts of complex eigenvectors provide the spatial patterns of oscillation.

Repeated Eigenvalues: When eigenvalues are repeated with insufficient eigenvectors, generalized eigenvectors and Jordan canonical form provide the complete solution structure.

Mechanical Applications: Normal mode analysis of vibrating systems demonstrates the power of eigenvalue methods in engineering applications. Each mode represents a fundamental pattern of oscillation.

Stability Analysis: Eigenvalue locations in the complex plane completely determine stability. This provides a systematic approach to analyzing system behavior without solving the equations explicitly.

Numerical Methods: Computational eigenvalue algorithms enable the analysis of large systems arising in practical applications. Understanding these methods is essential for modern scientific computing.

The eigenvalue approach provides both theoretical insight and computational tools that extend far beyond linear differential equations. These methods form the foundation for understanding more complex phenomena including bifurcations, chaos, and the behavior of nonlinear systems near equilibria.

Computational Companion: All eigenvalue computations, diagonalization procedures, and applications discussed in this lecture are implemented with detailed examples in `lecture4.py`. The code includes both analytical and numerical approaches to eigenvalue problems.