# Ordinary Differential Equations

Lecture Notes

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## 1 Lecture 3: Linear Systems and Matrix Methods

### 1.0.1 Introduction to Linear Systems

Linear systems of differential equations form the foundation for understanding more complex dynamical behavior. They arise naturally in many applications and provide the local approximation to nonlinear systems near equilibrium points. The complete solvability of linear systems makes them an essential stepping stone to understanding general dynamical systems.

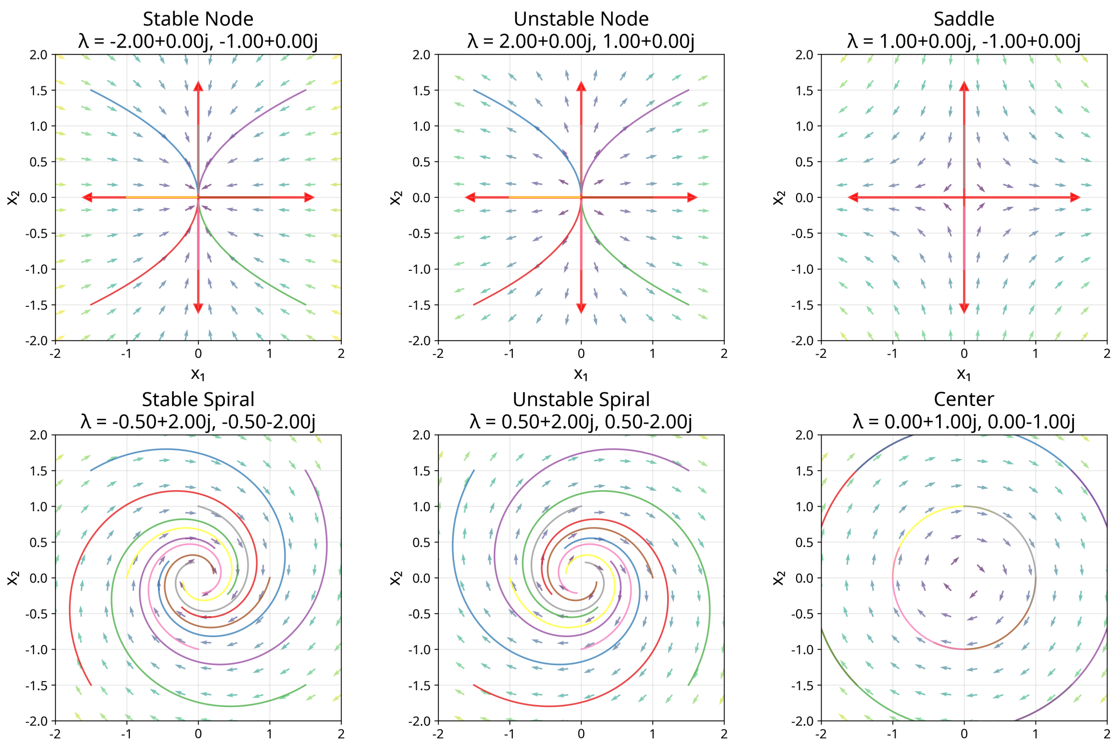


Figure 1: Classification of linear systems by eigenvalue type: stable/unstable nodes, saddles, spirals, and centers. Each panel shows phase portraits with eigenvector directions and typical trajectories.

A linear system of differential equations has the form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{A}(t)$ is an $n \times n$ matrix of coefficients, and $\mathbf{b}(t)$ is an $n$-dimensional forcing term.

When $\mathbf{b}(t) = \mathbf{0}$, the system is homogeneous:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} \tag{2}$$

The linearity of these systems means that the principle of superposition applies: if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions to the homogeneous system, then any linear combination $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is also a solution.

### 1.0.2 Constant Coefficient Systems

The most tractable case occurs when the coefficient matrix is constant: $\mathbf{A}(t) = \mathbf{A}$. This leads to the autonomous linear system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{3}$$

The solution to this system can be expressed using the matrix exponential:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 \tag{4}$$

where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial condition.

### 1.0.3 Matrix Exponential

The matrix exponential is defined by the convergent series:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} \tag{5}$$

This series converges for all finite $t$ and any matrix $\mathbf{A}$. The matrix exponential satisfies several important properties:

- $e^{\mathbf{0}} = \mathbf{I}$ (identity matrix)
- $\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$
- $e^{\mathbf{A}(t+s)} = e^{\mathbf{A}t}e^{\mathbf{A}s}$ (when $\mathbf{A}$ commutes with itself)
- $(e^{\mathbf{A}t})^{-1} = e^{-\mathbf{A}t}$

### 1.0.4 Computing the Matrix Exponential

While the series definition is theoretically important, practical computation of the matrix exponential typically uses eigenvalue decomposition or other numerical methods.

**Computational Note:** The file `lecture3.py` contains implementations for computing matrix exponentials using various methods, including eigenvalue decomposition, Padé approximation, and scaling and squaring algorithms.

### 1.0.5 Eigenvalue Analysis

The behavior of linear systems is fundamentally determined by the eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$.

3.1: Eigenvalues and Eigenvectors For an $n \times n$ matrix $\mathbf{A}$, a scalar $\lambda$ is an eigenvalue and a nonzero vector $\mathbf{v}$ is a corresponding eigenvector if:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{6}$$

The eigenvalues are the roots of the characteristic polynomial:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{7}$$

### 1.0.6 Real Distinct Eigenvalues

When $\mathbf{A}$ has $n$ real, distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, the general solution is:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n \tag{8}$$

The constants $c_1, c_2, \dots, c_n$ are determined by initial conditions.

3.1: Two-Dimensional System with Real Eigenvalues Consider the system:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\mathbf{x} \tag{9}$$

The characteristic equation is:

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0 \tag{10}$$

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = -1$

For $\lambda_1 = 4$: $(\mathbf{A} - 4\mathbf{I})\mathbf{v}_1 = \mathbf{0}$ gives $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

For $\lambda_2 = -1$: $(\mathbf{A} + \mathbf{I})\mathbf{v}_2 = \mathbf{0}$ gives $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General solution:

$$\mathbf{x}(t) = c_1 e^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{11}$$

### 1.0.7 Complex Eigenvalues

When $\mathbf{A}$ has complex eigenvalues, they occur in conjugate pairs for real matrices. If $\lambda = \alpha + i\beta$ is an eigenvalue with eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$, then the real solutions are:

$$\mathbf{x}_1(t) = e^{\alpha t}(\mathbf{u}\cos(\beta t) - \mathbf{w}\sin(\beta t)) \tag{12}$$

$$\mathbf{x}_2(t) = e^{\alpha t}(\mathbf{w}\cos(\beta t) + \mathbf{u}\sin(\beta t)) \tag{13}$$

These solutions represent spiraling motion in the phase plane, with the exponential factor $e^{\alpha t}$ determining whether the spiral converges to the origin ($\alpha < 0$), diverges from it ($\alpha > 0$), or maintains constant amplitude ($\alpha = 0$).

3.2: System with Complex Eigenvalues Consider the system:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{x} \tag{14}$$

The characteristic equation is:

$$\det \begin{pmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 + 4 = \lambda^2 + 2\lambda + 5 = 0 \tag{15}$$

Eigenvalues: $\lambda = -1 \pm 2i$

For $\lambda = -1 + 2i$, the eigenvector is $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Real solutions:

$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} \tag{16}$$

$$\mathbf{x}_2(t) = e^{-t} \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} \tag{17}$$

General solution:

$$\mathbf{x}(t) = e^{-t} \begin{pmatrix} c_1 \cos(2t) + c_2 \sin(2t) \\ -c_1 \sin(2t) + c_2 \cos(2t) \end{pmatrix} \tag{18}$$

### 1.0.8 Repeated Eigenvalues

When an eigenvalue has algebraic multiplicity greater than its geometric multiplicity, we need generalized eigenvectors to construct the complete solution.

For a repeated eigenvalue $\lambda$ with geometric multiplicity less than algebraic multiplicity, we find generalized eigenvectors $\mathbf{v}_k$ satisfying:

$$(\mathbf{A} - \lambda \mathbf{I})^k \mathbf{v}_k = \mathbf{0} \tag{19}$$

The corresponding solutions involve polynomial terms multiplied by exponentials.

### 1.0.9 Classification of Two-Dimensional Linear Systems

For two-dimensional systems, the qualitative behavior is completely determined by the eigenvalues of the coefficient matrix. The classification depends on the trace $\text{tr}(\mathbf{A}) = \lambda_1 + \lambda_2$ and determinant $\det(\mathbf{A}) = \lambda_1 \lambda_2$.

3.1: Classification of 2D Linear Systems For the system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with eigenvalues $\lambda_1, \lambda_2$:

**Real Distinct Eigenvalues:**

- **Stable Node:** $\lambda_1, \lambda_2 < 0$ (both negative)
- **Unstable Node:** $\lambda_1, \lambda_2 > 0$ (both positive)

- **Saddle Point:** $\lambda_1 < 0 < \lambda_2$ (opposite signs)

**Complex Eigenvalues ($\lambda = \alpha \pm i\beta$, $\beta \neq 0$):**

- **Stable Spiral:** $\alpha < 0$ (negative real part)
- **Unstable Spiral:** $\alpha > 0$ (positive real part)
- **Center:** $\alpha = 0$ (purely imaginary)

**Repeated Eigenvalues:**

- **Stable/Unstable Node:** Complete set of eigenvectors
- **Degenerate Node:** Incomplete set of eigenvectors

The trace-determinant plane provides a convenient way to visualize this classification. The parabola $(\text{tr}(\mathbf{A}))^2 = 4\det(\mathbf{A})$ separates regions with real eigenvalues from those with complex eigenvalues.

### 1.0.10 Fundamental Matrix and Wronskian

For the homogeneous system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, a fundamental matrix $\mathbf{\Phi}(t)$ is any $n \times n$ matrix whose columns are linearly independent solutions.

3.2: Fundamental Matrix A fundamental matrix $\mathbf{\Phi}(t)$ for the system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ satisfies:

$$\frac{d\mathbf{\Phi}}{dt} = \mathbf{A}\mathbf{\Phi} \tag{20}$$

The general solution can be written as:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} \tag{21}$$

where $\mathbf{c}$ is a constant vector determined by initial conditions.

The Wronskian of the fundamental matrix is:

$$W(t) = \det(\mathbf{\Phi}(t)) \tag{22}$$

3.2: Abel's Formula For the linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$, the Wronskian satisfies:

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \text{tr}(\mathbf{A}(s))ds\right) \tag{23}$$

For constant coefficient systems:

$$W(t) = W(0)e^{\text{tr}(\mathbf{A})t} \tag{24}$$

#### 1.0.11 Nonhomogeneous Linear Systems

The nonhomogeneous system has the form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b}(t) \tag{25}$$

3.3: Structure of General Solution The general solution to the nonhomogeneous system is:

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) \tag{26}$$

where $\mathbf{x}_h(t)$ is the general solution to the homogeneous system and $\mathbf{x}_p(t)$ is any particular solution to the nonhomogeneous system.

#### 1.0.12 Variation of Parameters

The method of variation of parameters provides a systematic approach to finding particular solutions.

3.4: Variation of Parameters Formula If $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system, then a particular solution is:

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t) \int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{b}(s)ds \tag{27}$$

For constant coefficient systems, this becomes:

$$\mathbf{x}_p(t) = e^{\mathbf{A}t} \int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{b}(s)ds \tag{28}$$

#### 1.0.13 Method of Undetermined Coefficients

When the forcing term $\mathbf{b}(t)$ has a special form (polynomial, exponential, sinusoidal, or combinations), the method of undetermined coefficients can be more efficient than variation of parameters.

3.3: Forced Harmonic Oscillator Consider the system:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ F_0 \cos(\Omega t) \end{pmatrix} \tag{29}$$

This represents a forced harmonic oscillator with natural frequency $\omega$ and driving frequency $\Omega$. The homogeneous solution is:

$$\mathbf{x}_h(t) = c_1 \begin{pmatrix} \cos(\omega t) \\ -\omega \sin(\omega t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\omega t) \\ \omega \cos(\omega t) \end{pmatrix} \tag{30}$$

For the particular solution, we try:

$$\mathbf{x}_p(t) = \begin{pmatrix} A\cos(\Omega t) + B\sin(\Omega t) \\ C\cos(\Omega t) + D\sin(\Omega t) \end{pmatrix} \tag{31}$$

Substituting and solving yields the particular solution, which exhibits resonance when $\Omega = \omega$.

### 1.0.14 Stability Theory

The stability of linear systems is completely determined by the eigenvalues of the coefficient matrix.

3.3: Stability Definitions For the linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$:

**Asymptotically Stable:** All eigenvalues have negative real parts. All solutions approach zero as $t \to \infty$.

**Stable (Lyapunov):** All eigenvalues have non-positive real parts, and those with zero real part are simple. Solutions remain bounded.

**Unstable:** At least one eigenvalue has positive real part. Some solutions grow without bound.

3.5: Stability Criterion for Linear Systems The linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ is:

- Asymptotically stable if and only if $\text{Re}(\lambda_i) < 0$ for all eigenvalues $\lambda_i$
- Stable if and only if $\text{Re}(\lambda_i) \leq 0$ for all eigenvalues, with simple eigenvalues on the imaginary axis
- Unstable if and only if $\text{Re}(\lambda_i) > 0$ for at least one eigenvalue $\lambda_i$

### 1.0.15 Applications

### 1.0.16 Mechanical Systems

Linear mechanical systems with small displacements lead naturally to linear differential equations. Consider a system of masses connected by springs and dampers.

The equations of motion for $n$ masses can be written as:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t) \tag{32}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, and $\mathbf{f}(t)$ is the external forcing.

Converting to first-order form with $\mathbf{x} = [\mathbf{q}^T, \dot{\mathbf{q}}^T]^T$:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t) \end{pmatrix} \tag{33}$$

### 1.0.17 Electrical Circuits

Linear electrical circuits with resistors, inductors, and capacitors lead to linear systems. Using Kirchhoff's laws, the circuit equations can be written in matrix form.

For a circuit with $n$ independent loops, the equations have the form:

$$\mathbf{L}\frac{d\mathbf{i}}{dt} + \mathbf{R}\mathbf{i} + \mathbf{Q}\mathbf{q} = \mathbf{v}(t) \tag{34}$$

where $\mathbf{i}$ is the vector of loop currents, $\mathbf{q}$ is the vector of charges, and the matrices represent inductance, resistance, and inverse capacitance effects.

#### 1.0.18 Population Dynamics

Linear population models arise when considering small perturbations around equilibrium populations or when interaction terms are linearized.

A general linear population model has the form:

$$\frac{d\mathbf{n}}{dt} = \mathbf{A}\mathbf{n} \tag{35}$$

where $\mathbf{n}(t)$ represents population densities and $\mathbf{A}$ contains birth rates, death rates, and migration coefficients.

The dominant eigenvalue (largest real part) determines the long-term growth rate of the total population.

#### 1.0.19 Numerical Methods for Linear Systems

While linear systems can be solved analytically, numerical methods are important for large systems and when coefficient matrices are known only approximately.

#### 1.0.20 Matrix Exponential Computation

Computing $e^{\mathbf{A}t}$ numerically requires careful consideration of accuracy and stability. Common methods include:

1. **Eigenvalue decomposition**: When $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, then $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$
2. **Padé approximation**: Rational approximation to the matrix exponential
3. **Scaling and squaring**: Use the identity $e^{\mathbf{A}t} = (e^{\mathbf{A}t/2^k})^{2^k}$

#### 1.0.21 Numerical Integration

Standard ODE solvers can be applied to linear systems, though specialized methods can exploit the linear structure for improved efficiency and accuracy.

**Computational Note:** The file `lecture3.py` includes comprehensive implementations of matrix exponential computation, eigenvalue analysis, and numerical methods specifically designed for linear systems.

This lecture has developed the complete theory for linear systems of differential equations:

**Matrix Exponential:** The solution $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$ provides the fundamental solution operator for linear systems. Understanding how to compute and interpret the matrix exponential is crucial for both theoretical analysis and practical computation.

**Eigenvalue Analysis:** The eigenvalues and eigenvectors of the coefficient matrix completely determine the qualitative behavior of linear systems. This provides a powerful tool for understanding stability, oscillations, and long-term behavior.

**Classification:** Two-dimensional linear systems can be completely classified based on their eigenvalues, leading to nodes, spirals, saddles, and centers. This classification extends to higher dimensions and provides the foundation for understanding nonlinear systems through linearization.

**Solution Methods:** Both homogeneous and nonhomogeneous linear systems can be solved systematically using eigenvalue methods, variation of parameters, and undetermined coefficients. These methods provide exact solutions that serve as benchmarks for numerical methods.

**Stability Theory:** Linear stability analysis provides the foundation for understanding the behavior of more complex systems. The connection between eigenvalues and stability is fundamental to dynamical systems theory.

**Applications:** Linear systems arise naturally in mechanical engineering, electrical circuits, population dynamics, and many other fields. The mathematical framework developed here applies broadly across science and engineering.

The techniques developed in this lecture provide the foundation for understanding nonlinear systems through linearization, which we will explore in subsequent lectures. The interplay between local linear analysis and global nonlinear behavior is a central theme in modern dynamical systems theory.

**Computational Companion:** All theoretical concepts, solution methods, and applications discussed in this lecture are implemented with detailed examples in `lecture3.py`. The code includes visualization tools for phase portraits, eigenvalue analysis, and stability regions.