Ordinary Differential Equations

Lecture Notes

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1 Lecture 1: Introduction and First-Order Equations

1.0.1 Introduction to Differential Equations

Differential equations form the mathematical foundation for describing change and evolution in natural and engineered systems. From the motion of planets to the dynamics of neural networks, from population growth to financial markets, differential equations provide the language for modeling how quantities vary with respect to time, space, or other independent variables.

An ordinary differential equation (ODE) is an equation involving an unknown function and its derivatives with respect to a single independent variable. The general form of an n-th order ODE is:

$$F\left(t, y, y', y'', \ldots, y^{(n)}\right) = 0 \tag{1}$$

where $y = y(t)$ is the unknown function, t is the independent variable (often representing time), and $y^{(k)}$ denotes the k-th derivative of y with respect to t.

The order of a differential equation is the highest derivative that appears in the equation. The degree is the power of the highest-order derivative when the equation is written as a polynomial in the derivatives. Most equations we encounter are first-degree, meaning they are linear in the highest derivative.

1.0.2 Classification of Differential Equations

Differential equations can be classified in several important ways:

Order: First-order equations involve only the first derivative, second-order equations involve up to the second derivative, and so forth. Higher-order equations often arise from physical principles involving acceleration (second derivatives) or higher-order effects.

Linearity: A differential equation is linear if it can be written in the form:

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t) \tag{2}$$

If $g(t) = 0$, the equation is homogeneous; otherwise, it is nonhomogeneous. Linear equations have the crucial property that linear combinations of solutions are also solutions (superposition principle).

Autonomy: An autonomous equation does not explicitly depend on the independent variable. For first-order equations, this means $\frac{dy}{dt} = f(y)$ rather than $\frac{dy}{dt} = f(t, y)$. Autonomous equations have special geometric properties that simplify their analysis.

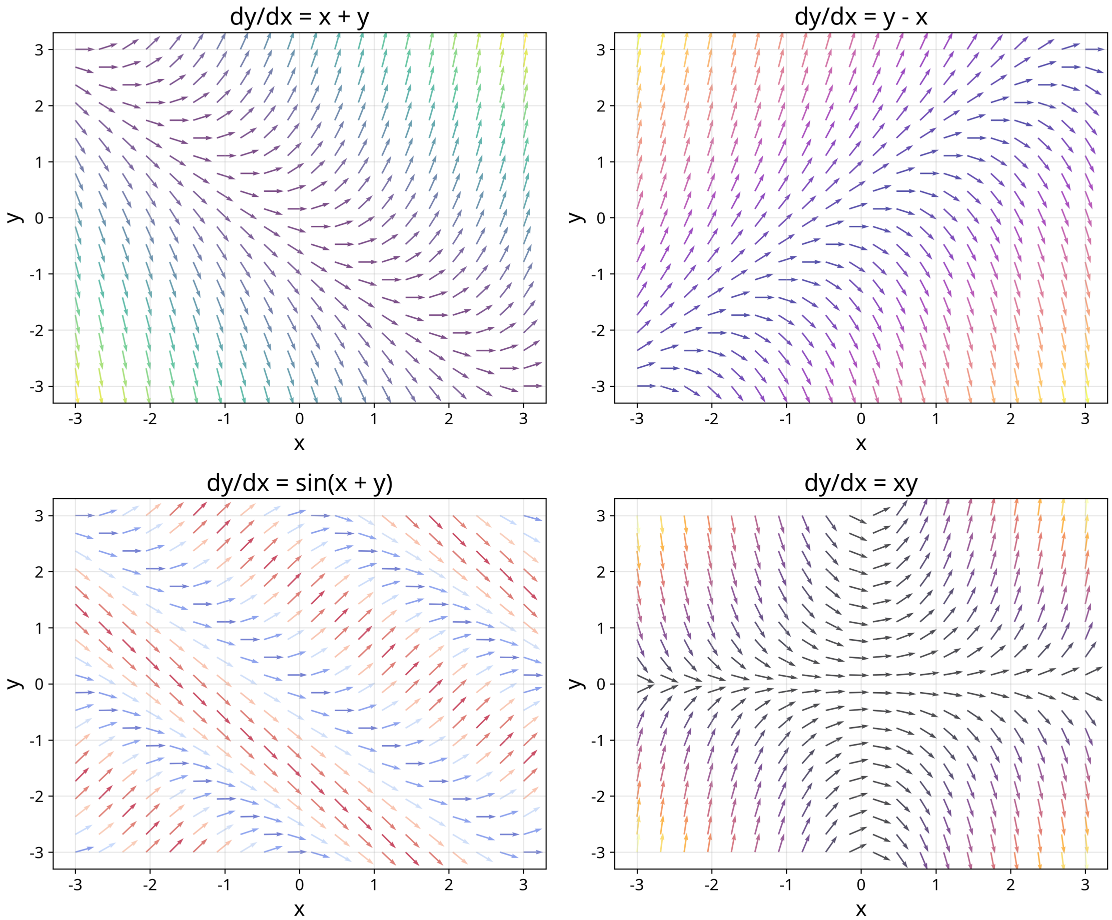

Figure 1: Direction fields for various first-order ODEs showing the geometric interpretation of differential equations. Each arrow indicates the slope dy/dx at that point, providing visual insight into solution behavior.

1.0.3 Existence and Uniqueness Theory

Before attempting to solve differential equations, we must understand when solutions exist and when they are unique. This fundamental question was answered by a series of theorems developed in the late 19th and early 20th centuries.

1.1: Picard-Lindelöf Theorem Consider the initial value problem:

$$\frac{dy}{dt} = f(t, y) \tag{3}$$

$$y(t_0) = y_0 \tag{4}$$

If $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R = \{(t, y) : |t - t_0| \leq a, |y - y_0| \leq b\}$, then there exists a positive number $h \leq \min(a, b/M)$ where $M = \max_{(t,y)\in R} |f(t, y)|$, such that the initial value problem has a unique solution $y(t)$ for $t \in [t_0 - h, t_0 + h]$.

This theorem guarantees both existence and uniqueness of solutions under reasonable continuity conditions. The proof, which we outline here, uses the method of successive approximations (Picard iteration).

Proof Outline: We convert the differential equation to an equivalent integral equation:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds \tag{5}$$

We then define a sequence of functions:

$$y_0(t) = y_0 \tag{6}$$

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\, ds \tag{7}$$

Under the given conditions, this sequence converges uniformly to the unique solution.

The geometric interpretation of this theorem is profound: through each point (t_0, y_0) in the domain where the conditions are satisfied, there passes exactly one solution curve. This means that solution curves cannot cross each other, a fact that has important implications for the qualitative behavior of solutions.

1.0.4 Failure of Uniqueness

When the conditions of the Picard-Lindelöf theorem are not met, uniqueness can fail dramatically.

1.1: Non-unique Solutions Consider the initial value problem:

$$\frac{dy}{dt} = 3y^{2/3} \tag{8}$$

$$y(0) = 0 \tag{9}$$

The function $f(t, y) = 3y^{2/3}$ is continuous everywhere, but $\frac{\partial f}{\partial y} = 2y^{-1/3}$ is not continuous at $y = 0$. This equation has infinitely many solutions:

$$y(t) = \begin{cases} 0 & \text{for } t \leq c \\ (t-c)^3 & \text{for } t > c \end{cases} \tag{10}$$

for any $c \geq 0$.

This example illustrates why the continuity of the partial derivative is crucial for uniqueness.

1.0.5 Geometric Interpretation: Direction Fields

One of the most powerful tools for understanding first-order differential equations is the geometric approach using direction fields (also called slope fields). This method provides visual insight into solution behavior without requiring explicit solutions.

For the differential equation $\frac{dy}{dt} = f(t, y)$, the direction field is constructed by drawing short line segments with slope $f(t, y)$ at each point (t, y) in the ty-plane. These segments indicate the direction that solution curves must follow at each point.

The construction process involves:

1. Choose a grid of points (t_i, y_j) in the region of interest 2. At each point, calculate the slope $m_{ij} = f(t_i, y_j)$ 3. Draw a short line segment through (t_i, y_j) with slope m_{ij} 4. The collection of all these segments forms the direction field

Solution curves are then tangent to the direction field at every point. This allows us to sketch approximate solutions by following the flow indicated by the direction field.

1.0.6 Isoclines

Isoclines are curves along which the direction field has constant slope. For the equation $\frac{dy}{dt} = f(t, y)$, the isocline corresponding to slope m is the curve defined by:

$$f(t, y) = m \tag{11}$$

Isoclines are particularly useful for sketching direction fields by hand, as they allow systematic construction of regions with similar slopes.

Computational Note: The file `lecture1.py` contains implementations for generating direction fields, plotting isoclines, and visualizing solution curves for various first-order equations.

1.0.7 Separable Equations

Separable equations form one of the most important classes of first-order ODEs that can be solved analytically. These equations have the special form:

$$\frac{dy}{dt} = g(t)h(y) \tag{12}$$

where the right-hand side can be factored into a function of t times a function of y.

1.0.8 Solution Method

The solution technique involves separating variables and integrating:

1. Rewrite the equation as: $\frac{dy}{h(y)} = g(t)dt$ 2. Integrate both sides: $\int \frac{dy}{h(y)} = \int g(t)dt + C$ 3. Solve for y if possible

This method works provided $h(y) \neq 0$ in the region of interest. Points where $h(y) = 0$ correspond to equilibrium solutions where $\frac{dy}{dt} = 0$.

1.2: Population Growth Model The logistic equation models population growth with limited resources:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{13}$$

where $P(t)$ is the population, r is the intrinsic growth rate, and K is the carrying capacity.
This is separable: $\frac{dP}{P(1-P/K)} = rdt$
Using partial fractions:

$$\frac{1}{P(1 - P/K)} = \frac{1}{P} + \frac{1/K}{1 - P/K} \tag{14}$$

Integrating and solving yields:

$$P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right)e^{-rt}} \tag{15}$$

where $P_0 = P(0)$ is the initial population.

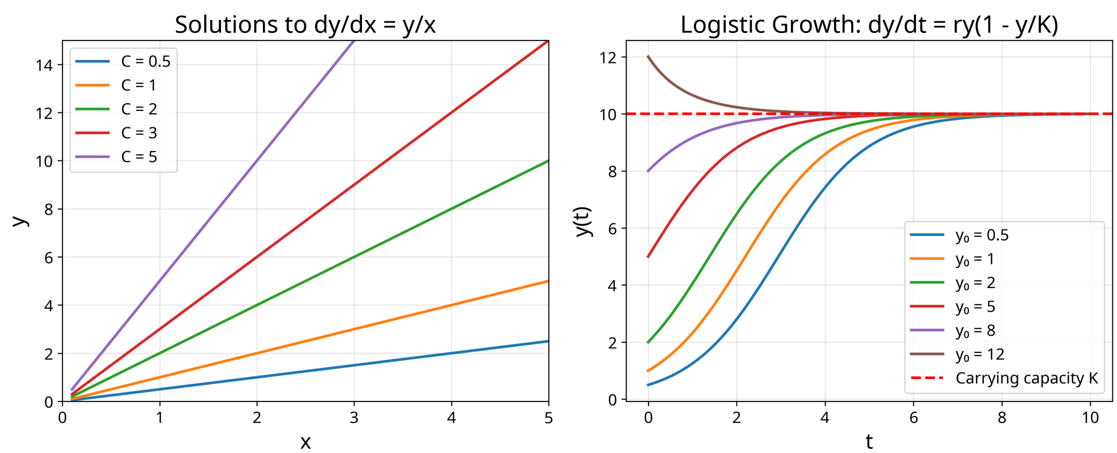

Figure 2: Solutions to separable equations: (left) Linear growth dy/dx = y/x with various initial conditions, (right) Logistic growth showing approach to carrying capacity K for different initial populations.

1.0.9 Equilibrium Solutions and Stability

Equilibrium solutions occur where $\frac{dy}{dt} = 0$. For separable equations $\frac{dy}{dt} = g(t)h(y)$, equilibria occur where $h(y) = 0$ (assuming $g(t) \neq 0$).

The stability of equilibria can be determined by examining the sign of $h(y)$ near equilibrium points:

- If $h(y) > 0$ for y slightly above the equilibrium and $h(y) < 0$ for y slightly below, the equilibrium is stable - If the signs are reversed, the equilibrium is unstable - If $h(y)$ has the same sign on both sides, the equilibrium is semi-stable

1.0.10 Linear First-Order Equations

Linear first-order equations have the standard form:

$$\frac{dy}{dt} + p(t)y = q(t) \tag{16}$$

These equations can always be solved using the integrating factor method, making them one of the most tractable classes of differential equations.

1.0.11 Integrating Factor Method

The key insight is to multiply the equation by an integrating factor $\mu(t)$ that makes the left side a perfect derivative:

1. Choose $\mu(t) = e^{\int p(t)dt}$ 2. Multiply the equation by $\mu(t)$: $\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)q(t)$ 3. Recognize that the left side is $\frac{d}{dt}[\mu(t)y]$ 4. Integrate: $\mu(t)y = \int \mu(t)q(t)dt + C$ 5. Solve for y: $y = \frac{1}{\mu(t)}\left(\int \mu(t)q(t)dt + C\right)$

1.3: RC Circuit An RC circuit with time-varying voltage source satisfies:

$$RC\frac{dV_C}{dt} + V_C = V_{in}(t) \tag{17}$$

where V_C is the capacitor voltage and $V_{in}(t)$ is the input voltage.

Rewriting in standard form: $\frac{dV_C}{dt} + \frac{1}{RC}V_C = \frac{V_{in}(t)}{RC}$

The integrating factor is $\mu(t) = e^{t/(RC)}$, leading to:

$$V_C(t) = e^{-t/(RC)} \left(V_C(0) + \frac{1}{RC} \int_0^t e^{s/(RC)} V_{in}(s) ds \right) \quad (18)$$

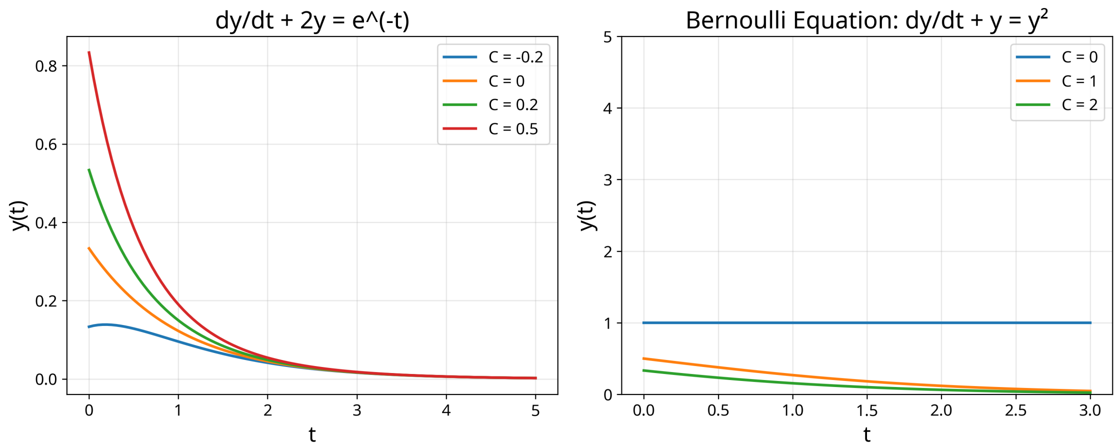

Figure 3: Solutions to linear first-order equations: (left) Integrating factor method example showing multiple solution curves, (right) Bernoulli equation solutions demonstrating nonlinear behavior that can be linearized through substitution.

1.0.12 Homogeneous vs. Nonhomogeneous Equations

When $q(t) = 0$, the equation is homogeneous: $\frac{dy}{dt} + p(t)y = 0$

The solution is simply: $y = Ce^{-\int p(t)dt}$

For the nonhomogeneous case, the general solution is the sum of: - The general solution to the homogeneous equation (complementary solution) - Any particular solution to the nonhomogeneous equation

This structure reflects the linearity of the equation and will be a recurring theme throughout our study of linear differential equations.

1.0.13 Applications and Modeling

Differential equations arise naturally in modeling dynamic processes across all areas of science and engineering. The key to successful modeling is identifying the fundamental principles that govern the system's behavior and translating them into mathematical relationships.

1.0.14 Newton's Law of Cooling

Newton's law of cooling states that the rate of temperature change is proportional to the temperature difference between an object and its environment:

$$\frac{dT}{dt} = -k(T - T_{env}) \quad (19)$$

where $T(t)$ is the object's temperature, T_{env} is the environmental temperature, and $k > 0$ is the cooling constant.

This is a linear first-order equation with solution:

$$T(t) = T_{env} + (T_0 - T_{env})e^{-kt} \tag{20}$$

The solution shows exponential decay toward the environmental temperature, with time constant $\tau = 1/k$.

1.0.15 Chemical Kinetics

Many chemical reactions follow first-order kinetics, where the reaction rate is proportional to the concentration of reactant:

$$\frac{d[A]}{dt} = -k[A] \tag{21}$$

This gives exponential decay: $[A](t) = [A]_0 e^{-kt}$

The half-life of the reaction is $t_{1/2} = \frac{\ln 2}{k}$, independent of the initial concentration.

1.0.16 Radioactive Decay

Radioactive decay follows the same mathematical model as first-order chemical kinetics:

$$\frac{dN}{dt} = -\lambda N \tag{22}$$

where $N(t)$ is the number of radioactive nuclei and λ is the decay constant.

The solution $N(t) = N_0 e^{-\lambda t}$ leads to the concept of half-life: $t_{1/2} = \frac{\ln 2}{\lambda}$.

1.0.17 Numerical Considerations

While analytical solutions provide exact answers and theoretical insight, many differential equations cannot be solved in closed form. Numerical methods bridge this gap by providing approximate solutions with controlled accuracy.

For first-order equations $\frac{dy}{dt} = f(t, y)$ with initial condition $y(t_0) = y_0$, the simplest numerical method is Euler's method:

$$y_{n+1} = y_n + h \cdot f(t_n, y_n) \tag{23}$$

where h is the step size and $y_n \approx y(t_n)$ with $t_n = t_0 + nh$.

Euler's method has a geometric interpretation: at each step, we follow the tangent line (given by the direction field) for a distance h. The accuracy depends on the step size, with smaller steps generally giving better approximations.

More sophisticated methods like Runge-Kutta achieve higher accuracy by evaluating the derivative at multiple points within each step.

Computational Note: The file `lecture1.py` includes implementations of Euler's method, improved Euler method, and fourth-order Runge-Kutta method, along with error analysis and comparison studies.

This first lecture has established the fundamental concepts that will guide our study of differential equations:

Existence and Uniqueness: The Picard-Lindelöf theorem provides conditions under which initial value problems have unique solutions. Understanding when solutions exist and are unique is crucial for both theoretical analysis and practical applications.

Geometric Perspective: Direction fields provide visual insight into solution behavior and help develop intuition about differential equations. This geometric viewpoint complements analytical methods and is especially valuable for nonlinear equations.

Analytical Methods: Separable equations and linear first-order equations represent important classes that can be solved exactly. The methods developed here—separation of variables and integrating factors—are fundamental techniques that extend to more complex situations.

Applications: Differential equations provide the mathematical framework for modeling change in natural and engineered systems. The examples of population growth, cooling, chemical kinetics, and radioactive decay illustrate how mathematical principles translate into practical insights.

Numerical Methods: When analytical solutions are not available, numerical methods provide approximate solutions. Understanding the relationship between analytical and numerical approaches is essential for modern scientific computing.

The concepts introduced in this lecture form the foundation for everything that follows. In the next lecture, we will extend these ideas to systems of first-order equations and explore the rich geometric structure that emerges in higher dimensions.

Computational Companion: All examples, visualizations, and numerical methods discussed in this lecture are implemented in `lecture1.py`. Students are encouraged to run and modify these examples to deepen their understanding of the theoretical concepts.