Lecture Series: Ordinary Differential Equations

Master of Science in Artificial Intelligence, Master of Science in Computational Science

Fall 2024

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Fall 2024

Contents

Lecture 1: Introduction to Differential Equations

Differential equations describe the relationship between an unknown function and its derivatives. They arise naturally in many fields of science, engineering, economics, and other areas where systems undergo change. Differential equations are essential for modeling real-world phenomena such as motion, population dynamics, chemical reactions, and electric circuits.

Definition

A *differential equation* (DE) is an equation that involves an unknown function $x(t)$ and its derivatives. The order of a differential equation is determined by the highest derivative present in the equation.

Example

Consider the first-order differential equation:

$$x'(t) = -x(t) + e^{-t}$$

A solution to this equation is $x(t) = te^{-t}$. This solution can be verified by substitution into the equation.

Here are four examples of differential equations that arise in various applications:

- $\theta'' + \sqrt{\frac{g}{l}} \sin\theta = 0$ (Pendulum equation): Models the angular deflections $\theta(t)$ of a pendulum of length l.
- $Rq' + \frac{1}{C}q = \sin\omega t$ (RC circuit equation): Models the charge $q(t)$ on a capacitor in an electrical circuit containing a resistor and a capacitor.
- $p' = rp\left(1 - \frac{p}{K}\right)$ (Population growth equation): Known as the logistic equation, this models the population $p(t)$ of an animal species in a closed environment.
- $T' = -h(T - Q)$ (Heating-cooling equation): Models the temperature $T(t)$ of an object placed in an environment of temperature Q.

1.1 First-Order Differential Equations

A first-order differential equation involves only the first derivative of the unknown function. These equations often arise from physical laws. For example, consider Newton's second law of motion for a particle of mass m subjected to a resistive force:

$$mv'(t) = -kv(t)$$

where $k > 0$ is a constant. The solution is:

$$v(t) = Ve^{-kt/m}$$

where V is the initial velocity at $t = 0$.

An *initial value problem* (IVP) specifies the value of the unknown function at an initial point, allowing us to find a unique solution to the differential equation. For example:

$$u'(t) = -u(t) + e^{-t}, \quad u(0) = 1$$

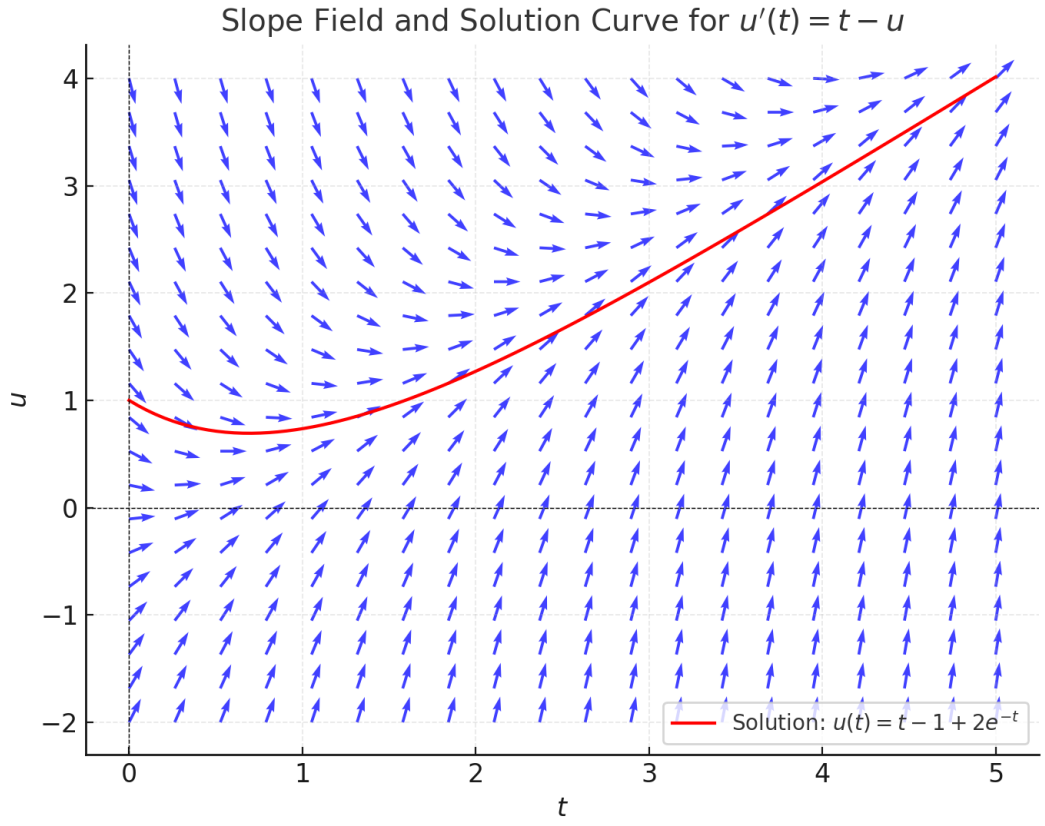
The solution to this IVP is:

$$u(t) = te^{-t}$$

The geometric interpretation of a differential equation involves visualizing its solutions through slope fields. A slope field represents the slopes of solutions at various points in the $t - x$ plane. This allows us to see the behavior of solutions without solving the equation analytically.

Example

Consider the differential equation $u'(t) = t - u$. The slope field can be plotted by computing the slopes u' at various points (t, u). Solution curves align with these slopes, providing a graphical representation of the solution.



A *pure time equation* is a differential equation where the right-hand side depends only on the independent variable t, not on the dependent variable u. These equations are easy to solve by direct integration.

Definition

A *pure time equation* is of the form:

$$u'(t) = g(t)$$

The solution is obtained by integrating both sides:

$$u(t) = \int g(t)\,dt + C$$

where C is an arbitrary constant.

Example

Solve the pure time equation:

$$u'(t) = t^2 - 1$$

The solution is found by integrating:

$$u(t) = \frac{t^3}{3} - t + C$$

1.2 Growth and Decay Models

A fundamental model for many natural processes is exponential growth or decay. The general form of an exponential decay equation is:

$$x'(t) = -rx$$

where $r > 0$ is a constant. The solution is:

$$x(t) = x_0 e^{-rt}$$

where x_0 is the initial value at $t = 0$.

Example

Consider the radioactive decay problem where a 4-gram sample decays to 0.8 grams in 3 years. The decay equation is:

$$x'(t) = -rx, \quad x(0) = 4$$

The solution is:

$$x(t) = 4e^{-rt}$$

Using $x(3) = 0.8$, we solve for r:

$$r = -\frac{1}{3}\ln\left(\frac{0.8}{4}\right) = 0.536$$

The half-life is given by solving $x(\tau) = 2$:

$$2 = 4e^{-0.536\tau} \quad \Rightarrow \quad \tau = 1.293 \text{ years}$$

A graphical method for understanding the solutions of first-order differential equations is to plot their *slope fields* or *direction fields*. A slope field is a grid of points where short line segments represent the slope of the solution curve passing through that point.

Example

The slope field for the equation $x'(t) = -x + 2t$ can be computed by evaluating the right-hand side at various points in the $t - x$ plane. Using these values, we can sketch the solution curves that fit the slope field.

Nullclines and isoclines can also be used to study the qualitative behavior of solutions. Nullclines represent regions where the slope is zero, while isoclines represent curves where the slope is constant.

Example

Consider the equation $x'(t) = -tx + x^2$. The nullclines occur where $x'(t) = 0$, giving $x = 0$ and $x = t$. These lines divide the plane into regions with different signs of the slope.

1.3 Antiderivatives and Integration

In some cases, solving a differential equation reduces to finding an antiderivative. Given an equation $x'(t) = g(t)$, we can solve by integrating both sides:

$$x(t) = \int g(t)dt + C$$

Example

Solve the equation:

$$x'(t) = t^2 - 1, \quad x(1) = 2$$

Integrating:

$$x(t) = \frac{t^3}{3} - t + C$$

Using $x(1) = 2$, we find $C = \frac{8}{3}$, so the solution is:

$$x(t) = \frac{t^3}{3} - t + \frac{8}{3}$$

For higher-order equations, we can integrate successively to find the solution.

Example

For equations of the form $x'' = g(t)$, we can take two successive antiderivatives to find the general solution. Consider the DE

$$x'' = t + 2.$$

Then, integrating once, we get

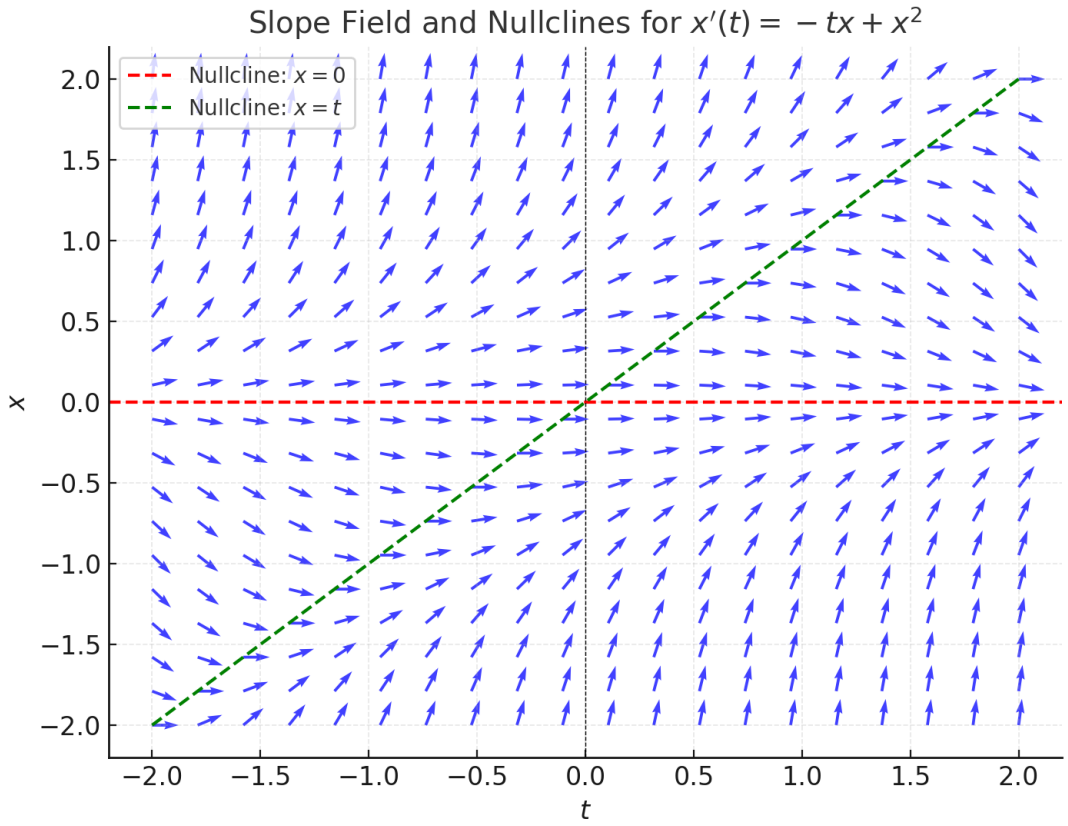
$$x' = \frac{1}{2}t^2 + 2t + C_1.$$

Integrating again,

$$x = \frac{1}{6}t^3 + t^2 + C_1 t + C_2.$$

Here C_1 and C_2 are two arbitrary constants. For second-order equations, we always expect two arbitrary constants, or a two-parameter family of solutions. It takes two initial conditions to determine the arbitrary constants. For example, if $x(0) = 1$ and if $x'(0) = 0$, then $C_1 = 0$ and $C_2 = 1$, and we obtain the particular solution

$$x(t) = \frac{1}{6}t^3 + t^2 + 1.$$



1.4 Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** plays a vital role in connecting differentiation and integration. This powerful theorem guarantees that, under certain conditions, integration can reverse the process of differentiation. As a result, it provides an essential tool for solving differential equations by allowing us to find the antiderivative of a function.

Theorem

Fundamental Theorem of Calculus.
Let f be a continuous real-valued function defined on the closed interval $[a, b]$, and let F be an antiderivative of f, meaning $F'(x) = f(x)$ for all x in $[a, b]$. Then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This theorem is crucial in solving differential equations, as it guarantees that integration can reverse differentiation, allowing us to find solutions to differential equations through integration. In some cases, the antiderivative may not have a closed form, and numerical methods or special functions are required.

Lecture 2: Separable Equations and Linear Equations

2.1 Separable Equations

A separable equation is a differential equation of the form:

$$\frac{dx}{dt} = f(x)g(t),$$

where the variables can be separated. To solve such an equation, we rearrange it so that all terms involving x are on one side and all terms involving t are on the other side. Once separated, we can integrate both sides.

Definition

A differential equation is said to be separable if it can be written as:

$$\frac{1}{f(x)}\,dx = g(t)\,dt.$$

To solve a separable equation of the form:

$$\frac{dx}{dt} = f(x)g(t),$$

we first separate the variables:

$$\frac{1}{f(x)}\,dx = g(t)\,dt.$$

Then, integrate both sides:

$$\int \frac{1}{f(x)}\,dx = \int g(t)\,dt + C,$$

where C is an arbitrary constant of integration. This provides the general solution of the differential equation.

Example

Solve the separable equation:

$$\frac{dx}{dt} = xt.$$

Rewriting it in separable form:

$$\frac{1}{x}\,dx = t\,dt.$$

Now, integrate both sides:

$$\ln|x| = \frac{t^2}{2} + C.$$

Exponentiating both sides gives:

$$x(t) = C_1 e^{\frac{t^2}{2}},$$

where $C_1 = e^C$ is an arbitrary constant.

Example

Consider the population growth equation:

$$\frac{dp}{dt} = rp,$$

where r is a constant representing the growth rate. This equation is separable, and we can solve it by writing:

$$\frac{1}{p}\,dp = r\,dt.$$

Integrating both sides:

$$\ln|p| = rt + C,$$

we find:

$$p(t) = C_1 e^{rt}.$$

This represents exponential growth or decay, depending on the sign of r.

2.2 Linear Equations

A linear first-order differential equation has the form:

$$\frac{dx}{dt} + p(t)x = q(t).$$

To solve such an equation, we use an integrating factor.

Definition

An integrating factor is a function $\mu(t)$ such that when both sides of the differential equation are multiplied by $\mu(t)$, the left-hand side becomes the derivative of a product:

$$\mu(t) = e^{\int p(t)\,dt}.$$

Given a linear differential equation $\frac{dx}{dt} + p(t)x = q(t)$, multiply both sides by the integrating factor:

$$\mu(t) = e^{\int p(t)\,dt}.$$

This transforms the equation into:

$$\frac{d}{dt}\left(\mu(t)x(t)\right) = \mu(t)q(t).$$

We then integrate both sides to find the general solution:

$$\mu(t)x(t) = \int \mu(t)q(t)\,dt + C.$$

Example

Solve the equation:

$$\frac{dx}{dt} + 2x = t^2.$$

First, identify $p(t) = 2$. The integrating factor is:

$$\mu(t) = e^{\int 2\,dt} = e^{2t}.$$

Multiplying through by $\mu(t)$:

$$e^{2t}\frac{dx}{dt} + 2e^{2t}x = e^{2t}t^2.$$

The left-hand side is now the derivative of $e^{2t}x$, so:

$$\frac{d}{dt}\left(e^{2t}x\right) = e^{2t}t^2.$$

Integrating both sides:

$$e^{2t}x = \int e^{2t}t^2\,dt.$$

Using integration by parts, we find:

$$e^{2t}x = \frac{t^2}{2}e^{2t} - te^{2t} + \frac{1}{2}e^{2t} + C.$$

Solving for $x(t)$, the general solution is:

$$x(t) = \frac{t^2}{2} - t + \frac{1}{2} + Ce^{-2t}.$$

Lecture 3: Dynamical Systems

3.1 One-Dimensional Dynamical Systems

An autonomous first-order differential equation is a system where the right-hand side depends only on the dependent variable x, and not explicitly on the independent variable t. These equations are often written as:

$$\frac{dx}{dt} = f(x).$$

3.1.1 Equilibrium Points

Equilibrium points are values of x where $f(x) = 0$. At these points, the system does not change over time.

Definition

An equilibrium point x_e of the differential equation $\frac{dx}{dt} = f(x)$ is a value of x such that $f(x_e) = 0$.

The stability of an equilibrium point x_e can be analyzed by considering the sign of $f'(x_e)$:

- If $f'(x_e) < 0$, the equilibrium is *stable*, meaning solutions near x_e will move toward x_e.
- If $f'(x_e) > 0$, the equilibrium is *unstable*, meaning solutions near x_e will move away from x_e.

Example

Consider the equation:

$$\frac{dx}{dt} = x(1 - x).$$

The equilibrium points occur where $f(x) = 0$, which gives $x = 0$ and $x = 1$. To analyze the stability, compute $f'(x) = 1 - 2x$:

- At $x = 0$, $f'(0) = 1$, so $x = 0$ is an unstable equilibrium.
- At $x = 1$, $f'(1) = -1$, so $x = 1$ is a stable equilibrium.

3.1.2 Bifurcations

A bifurcation occurs when a small change in a parameter of a system causes a qualitative change in the system's behavior. Consider the equation:

$$\frac{dx}{dt} = r - x^2,$$

where r is a parameter. The equilibrium points depend on r, and their stability changes as r varies.

Example

For the equation $\frac{dx}{dt} = r - x^2$, the equilibrium points are found by solving:

$$r - x^2 = 0 \quad \Rightarrow \quad x = \pm\sqrt{r}.$$

For $r > 0$, there are two equilibrium points at $x = \pm\sqrt{r}$. The point $x = \sqrt{r}$ is stable, while $x = -\sqrt{r}$ is unstable. For $r = 0$, there is a single equilibrium point at $x = 0$, and for $r < 0$, there are no real equilibrium points.

Let x_e be an equilibrium point of the system $\frac{dx}{dt} = f(x)$, where $f(x_e) = 0$. The stability of x_e is determined by the sign of $f'(x_e)$:

- If $f'(x_e) < 0$, the equilibrium is *stable*, and small deviations from x_e will result in the system returning to x_e.

- If $f'(x_e) > 0$, the equilibrium is *unstable*, and small deviations from x_e will result in the system moving away from x_e.

Example

Let's analyze the stability of equilibrium points for the differential equation:

$$\frac{dx}{dt} = x(1 - x).$$

This equation describes a system with equilibrium points at $x = 0$ and $x = 1$.

Step 1: Find the equilibrium points. To find the equilibrium points, we set $f(x) = 0$:

$$x(1 - x) = 0.$$

This gives two equilibrium points:

$$x = 0 \quad \text{and} \quad x = 1.$$

Step 2: Compute the first derivative. Now, we take the derivative of $f(x) = x(1-x)$ to analyze the stability at the equilibrium points:

$$f'(x) = 1 - 2x.$$

Step 3: Analyze the stability at each equilibrium point.

- **At $x = 0$:**

  $$f'(0) = 1 - 2(0) = 1.$$

  Since $f'(0) > 0$, the equilibrium point at $x = 0$ is *unstable*. This means that small perturbations around $x = 0$ will cause the system to move away from $x = 0$.

- **At $x = 1$:**

  $$f'(1) = 1 - 2(1) = -1.$$

  Since $f'(1) < 0$, the equilibrium point at $x = 1$ is *stable*. This means that small perturbations around $x = 1$ will cause the system to return to $x = 1$.

Stability Analysis of Equilibrium Point

To understand why the sign of $f'(x_e)$ determines stability, consider the behavior near the equilibrium point x_e in two different scenarios:

- **Case 1:** $f'(x_e) < 0$ **(Negative Derivative)**
  - **Above** x_e **($x > x_e$):** Since $f(x) < 0$, we have $\frac{dx}{dt} < 0$, which causes x to decrease, driving the system back towards x_e.
  - **Below** x_e **($x < x_e$):** Since $f(x) > 0$, we have $\frac{dx}{dt} > 0$, which causes x to increase, also moving the system back towards x_e.

  In both cases, the system is driven back towards the equilibrium point x_e, indicating that it is a *stable* equilibrium.

- **Case 2:** $f'(x_e) > 0$ **(Positive Derivative)**
  - **Above** x_e **($x > x_e$):** Since $f(x) > 0$, we have $\frac{dx}{dt} > 0$, causing x to increase further, moving away from x_e.
  - **Below** x_e **($x < x_e$):** Since $f(x) < 0$, we have $\frac{dx}{dt} < 0$, causing x to decrease further, again moving away from x_e.

  In both scenarios, the system moves away from the equilibrium point x_e, which means it is an *unstable* equilibrium.

Thus, the sign of $f'(x_e)$ determines whether small perturbations are corrected (leading to stability) or amplified (leading to instability).

3.2 Existence and Uniqueness Theorem

The existence and uniqueness theorem provides a fundamental result regarding the solutions of first-order differential equations. It ensures that, under certain conditions, there exists a unique solution to the initial value problem.

Theorem

Consider the initial value problem:

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0.$$

If $f(t, x)$ and $\frac{\partial f}{\partial x}$ are continuous in a region around (t_0, x_0), then there exists a unique solution to the initial value problem in some interval around t_0.

Example

Consider the initial value problem:

$$\frac{dx}{dt} = t + x, \quad x(0) = 1.$$

The function $f(t, x) = t + x$ and its partial derivative with respect to x, $\frac{\partial f}{\partial x} = 1$, are continuous everywhere. Therefore, by the existence and uniqueness theorem, there exists a unique solution to this problem for all t.

This theorem guarantees that, for a broad class of differential equations, we can find unique solutions that are continuous and valid within some interval.

3.3 Homogeneous Second-Order Linear Equations

Second-order linear differential equations are fundamental in modeling a wide range of physical systems. They often arise in the study of mechanical vibrations, electrical circuits, population models, and other dynamic systems. The general form of a second-order linear homogeneous differential equation is:

$$ax''(t) + bx'(t) + cx(t) = 0$$

where a, b, and c are constants, and $x(t)$ is the unknown function. The behavior of the solutions depends on the nature of the coefficients and the system being modeled.

Consider a simple spring-mass system where a mass m is attached to a spring with stiffness k, and there is no external force or damping. Newton's second law gives the governing equation of motion as:

$$mx''(t) + kx(t) = 0,$$

where $x(t)$ is the displacement from equilibrium, m is the mass, and k is the spring constant. This system oscillates around its equilibrium point, and the solutions describe simple harmonic motion. We will explore the general methods for solving such equations in this lecture.

Theorem

General Solution for Homogeneous Second-Order Equations
The general solution of the second-order linear homogeneous differential equation:

$$ax''(t) + bx'(t) + cx(t) = 0$$

is determined by the roots of the associated characteristic equation:

$$a\lambda^2 + b\lambda + c = 0.$$

There are three possible cases for the roots of the characteristic equation:

- **Case 1:** Two distinct real roots λ_1 and λ_2
- **Case 2:** One repeated real root λ
- **Case 3:** Two complex conjugate roots $\lambda = \alpha \pm i\beta$

Each case gives rise to a different form of the general solution, which we will explore in the following sections.

To solve the equation $ax''(t) + bx'(t) + cx(t) = 0$, we assume a solution of the form $x(t) = e^{\lambda t}$, where λ is a constant. Substituting this assumed solution into the differential equation, we obtain:

$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0.$$

Since $e^{\lambda t} \neq 0$, we can divide both sides by $e^{\lambda t}$, leaving us with the **characteristic equation**:

$$a\lambda^2 + b\lambda + c = 0.$$

This is a quadratic equation in λ, and its solutions depend on the discriminant:

$$\Delta = b^2 - 4ac.$$

The form of the general solution depends on the nature of the roots of the characteristic equation, as stated in the theorem above.

When $\Delta > 0$, the characteristic equation has two distinct real roots λ_1 and λ_2. In this case, the general solution to the differential equation is:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

This form of the solution describes systems that may exponentially grow or decay, depending on the values of λ_1 and λ_2.

Example

Example: Real and Distinct Roots
Solve the differential equation:

$$x'' - 5x' + 6x = 0.$$

The characteristic equation is:

$$\lambda^2 - 5\lambda + 6 = 0$$

which factors as:

$$(\lambda - 3)(\lambda - 2) = 0.$$

Thus, the roots are $\lambda_1 = 3$ and $\lambda_2 = 2$. Therefore, the general solution is:

$$x(t) = c_1 e^{3t} + c_2 e^{2t}.$$

When $\Delta = 0$, the characteristic equation has a repeated real root $\lambda_1 = \lambda_2 = \lambda$. In this case, the general solution takes the form:

$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$

This occurs because the repeated root provides only one independent solution, and the second linearly independent solution must involve a factor of t.

Example

Example: Repeated Real Roots
Solve the equation:

$$x'' - 4x' + 4x = 0.$$

The characteristic equation is:

$$\lambda^2 - 4\lambda + 4 = 0$$

which simplifies to:

$$(\lambda - 2)^2 = 0.$$

Thus, the root is $\lambda = 2$, and the general solution is:

$$x(t) = (c_1 + c_2 t)e^{2t}.$$

When $\Delta < 0$, the characteristic equation has complex conjugate roots $\lambda = \alpha \pm i\beta$. In this case, the general solution is:

$$x(t) = e^{\alpha t} \left(c_1 \cos(\beta t) + c_2 \sin(\beta t) \right).$$

This form of the solution describes oscillatory behavior, where α determines the exponential growth or decay and β determines the frequency of oscillation.

Example

Example: Complex Roots
Solve the equation:

$$x'' + 2x' + 5x = 0.$$

The characteristic equation is:

$$\lambda^2 + 2\lambda + 5 = 0.$$

Using the quadratic formula, the roots are:

$$\lambda = -1 \pm 2i.$$

Thus, the general solution is:

$$x(t) = e^{-t} \left(c_1 \cos(2t) + c_2 \sin(2t) \right).$$

3.4 Applications to Classical Mechanics

Second-order linear differential equations frequently arise in mechanical systems, such as spring-mass-damper systems. These systems are governed by equations of the form:

$$mx'' + \gamma x' + kx = 0$$

where m is the mass, γ is the damping coefficient, and k is the spring constant.

Simple Harmonic Motion

When there is no damping ($\gamma = 0$), the system exhibits simple harmonic motion, and the equation becomes:

$$mx'' + kx = 0$$

The solution is oscillatory, with a natural frequency $\omega = \sqrt{\frac{k}{m}}$.

Example

Consider a mass-spring system with $m = 1$ kg and $k = 9$ N/m. The equation of motion is:

$$x'' + 9x = 0$$

The characteristic equation is:

$$\lambda^2 + 9 = 0$$

which has roots $\lambda = \pm 3i$. The general solution is:

$$x(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

Lecture 4: Homogeneous Second-Order Linear Equations

Second-order linear differential equations are fundamental in modeling a wide range of physical systems. They often arise in the study of mechanical vibrations, electrical circuits, population models, and other dynamic systems. The general form of a second-order linear homogeneous differential equation is:

$$ax''(t) + bx'(t) + cx(t) = 0$$

where a, b, and c are constants, and $x(t)$ is the unknown function. The behavior of the solutions depends on the nature of the coefficients and the system being modeled.

Consider a simple spring-mass system where a mass m is attached to a spring with stiffness k, and there is no external force or damping. Newton's second law gives the governing equation of motion as:

$$mx''(t) + kx(t) = 0,$$

where $x(t)$ is the displacement from equilibrium, m is the mass, and k is the spring constant. This is a second-order linear homogeneous differential equation of the form:

$$mx''(t) + 0 \cdot x'(t) + kx(t) = 0.$$

This system oscillates around its equilibrium point, and the solutions describe simple harmonic motion.

4.1 Characteristic Equation and General Solution

To solve the equation $ax''(t) + bx'(t) + cx(t) = 0$, we assume a solution of the form $x(t) = e^{\lambda t}$, where λ is a constant. Substituting this assumed solution into the differential equation, we obtain:

$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0.$$

Since $e^{\lambda t} \neq 0$, we can divide both sides by $e^{\lambda t}$, leaving us with the **characteristic equation**:

$$a\lambda^2 + b\lambda + c = 0.$$

This is a quadratic equation in λ, and its solutions depend on the discriminant:

$$\Delta = b^2 - 4ac.$$

The form of the general solution depends on the nature of the roots of the characteristic equation, as stated in the theorem below.

Theorem

General Solution for Homogeneous Second-Order Equations
Let a, b, c be constants and consider the second-order homogeneous linear differential equation:

$$ax''(t) + bx'(t) + cx(t) = 0.$$

Then, the general solution of the equation depends on the roots of the characteristic equation:

$$a\lambda^2 + b\lambda + c = 0.$$

There are three cases:

- **Case 1: Two distinct real roots.** If the discriminant $\Delta = b^2 - 4ac > 0$, the characteristic equation has two distinct real roots λ_1 and λ_2. The general solution is:
$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$
- **Case 2: One repeated real root.** If $\Delta = 0$, the characteristic equation has one real repeated root λ. The general solution is:
$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$
- **Case 3: Complex conjugate roots.** If $\Delta < 0$, the characteristic equation has complex conjugate roots $\lambda = \alpha \pm i\beta$. The general solution is:
$$x(t) = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right).$$

Example

Solve the differential equation:

$$x'' - 5x' + 6x = 0.$$

The characteristic equation is:

$$\lambda^2 - 5\lambda + 6 = 0$$

which factors as:

$$(\lambda - 3)(\lambda - 2) = 0.$$

Thus, the roots are $\lambda_1 = 3$ and $\lambda_2 = 2$. Therefore, the general solution is:

$$x(t) = c_1 e^{3t} + c_2 e^{2t}.$$

Example

Consider a damped mechanical oscillator governed by the equation:

$$x'' + 3x' + 2x = 0.$$

The characteristic equation is:

$$\lambda^2 + 3\lambda + 2 = 0$$

which factors as:

$$(\lambda + 1)(\lambda + 2) = 0.$$

Thus, the roots are $\lambda_1 = -1$ and $\lambda_2 = -2$. The general solution is:

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

This describes an exponentially decaying motion, which is typical for a damped system.
When $\Delta = 0$, the characteristic equation has a repeated real root $\lambda_1 = \lambda_2 = \lambda$. In this case, the general solution takes the form:

$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$

Theorem

Solution for Repeated Roots Let $ax''(t) + bx'(t) + cx(t) = 0$ have a repeated root λ for the characteristic equation $a\lambda^2 + b\lambda + c = 0$. Then the general solution is:

$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$

Example

Solve the equation:

$$x'' - 4x' + 4x = 0.$$

The characteristic equation is:

$$\lambda^2 - 4\lambda + 4 = 0$$

which simplifies to:

$$(\lambda - 2)^2 = 0.$$

Thus, the root is $\lambda = 2$, and the general solution is:

$$x(t) = (c_1 + c_2 t)e^{2t}.$$

Example

In a critically damped mechanical system, the equation of motion takes the form:

$$x'' + 4x' + 4x = 0.$$

The characteristic equation is:

$$\lambda^2 + 4\lambda + 4 = 0$$

which gives the repeated root $\lambda = -2$. Thus, the general solution is:

$$x(t) = (c_1 + c_2 t)e^{-2t}.$$

This represents a critically damped system, where the system returns to equilibrium without oscillating.

When $\Delta < 0$, the characteristic equation has complex conjugate roots $\lambda = \alpha \pm i\beta$. In this case, the general solution is:

$$x(t) = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right).$$

Example

Solve the equation:

$$x'' + 2x' + 5x = 0.$$

The characteristic equation is:

$$\lambda^2 + 2\lambda + 5 = 0.$$

Using the quadratic formula, the roots are:

$$\lambda = -1 \pm 2i.$$

Thus, the general solution is:

$$x(t) = e^{-t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right).$$

This solution represents an underdamped harmonic oscillator, where the system oscillates while slowly decaying over time.

Example

Consider the equation:

$$x'' + 9x = 0.$$

The characteristic equation is:

$$\lambda^2 + 9 = 0$$

which gives the complex roots $\lambda = \pm 3i$. The general solution is:

$$x(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

This represents simple harmonic motion, with a frequency of oscillation $\omega = 3$.

Lecture 5: Nonhomogeneous Second-Order Linear Equations

In this lecture, we will explore the solution of second-order linear differential equations of the form:

$$ax''(t) + bx'(t) + cx(t) = f(t),$$

where $f(t)$ is a non-zero function, representing an external force or input to the system. These equations are called **nonhomogeneous** because of the presence of $f(t)$, unlike the homogeneous case where $f(t) = 0$. We will study methods to find a particular solution for nonhomogeneous equations and discuss the phenomenon of resonance.

5.1 General Solution

We are dealing with the nonhomogeneous second-order linear differential equation:

$$ax''(t) + bx'(t) + cx(t) = f(t).$$

The solution to this equation can be written as the sum of two parts:

$$x(t) = x_h(t) + x_p(t),$$

where:

- $x_h(t)$ is the general solution to the corresponding homogeneous equation:

$$ax_h''(t) + bx_h'(t) + cx_h(t) = 0,$$

- $x_p(t)$ is a particular solution to the nonhomogeneous equation.

First, consider the homogeneous differential equation:

$$ax_h''(t) + bx_h'(t) + cx_h(t) = 0.$$

The general solution to this equation is:

$$x_h(t) = c_1x_1(t) + c_2x_2(t),$$

where $x_1(t)$ and $x_2(t)$ are two linearly independent solutions, and c_1, c_2 are arbitrary constants.

Now, consider the nonhomogeneous equation:

$$ax''(t) + bx'(t) + cx(t) = f(t),$$

where $f(t)$ is a known function. We seek a particular solution $x_p(t)$ that satisfies this equation.

The general solution to the nonhomogeneous equation is the sum of the homogeneous solution and the particular solution:

$$x(t) = x_h(t) + x_p(t).$$

To verify, substitute $x(t) = x_h(t) + x_p(t)$ into the nonhomogeneous equation:

$$a(x_h + x_p)'' + b(x_h + x_p)' + c(x_h + x_p) = f(t),$$

which expands as:

$$ax_h'' + bx_h' + cx_h + ax_p'' + bx_p' + cx_p = f(t).$$

Since $x_h(t)$ satisfies the homogeneous equation, $ax_h'' + bx_h' + cx_h = 0$, this simplifies to:

$$ax_p'' + bx_p' + cx_p = f(t),$$

confirming that $x(t) = x_h(t) + x_p(t)$ is the general solution.

We can define the linear differential operator L as:

$$L(x) = ax''(t) + bx'(t) + cx(t),$$

so that:

$$L(x_h) = 0 \quad \text{and} \quad L(x_p) = f(t).$$

5.2 Method of Undetermined Coefficients

This method is suitable when the nonhomogeneous term $f(t)$ is a simple function such as a polynomial, exponential, sine, or cosine. We make an educated guess for the form of the particular solution $x_p(t)$ based on the form of $f(t)$, and then solve for unknown coefficients by substituting the guessed solution into the original equation.

Given the equation:

$$ax'' + bx' + cx = f(t),$$

we proceed as follows:

1. Solve the homogeneous equation to find $x_h(t)$.

2. Make a reasonable guess for the form of $x_p(t)$, ensuring it resembles the form of $f(t)$. The guess should include unknown constants to be determined.

3. Substitute $x_p(t)$ into the original differential equation and solve for the unknown constants.

Example

Solve the equation:

$$x'' + 4x = \cos(2t).$$

Step 1: Solve the homogeneous equation $x'' + 4x = 0$. The characteristic equation is:

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda = \pm 2i.$$

Thus, the homogeneous solution is:

$$x_h(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

To find $x_p(t)$, we observe that $f(t) = \cos(2t)$. Since the homogeneous solution already contains terms involving $\cos(2t)$, we multiply by t to avoid duplication. Guess:

$$x_p(t) = At \sin(2t) + Bt \cos(2t).$$

Substitute $x_p(t)$ into the original equation:

$$(At \sin(2t) + Bt \cos(2t))'' + 4(At \sin(2t) + Bt \cos(2t)) = \cos(2t).$$

After differentiating and equating terms, solve for A and B, yielding the particular solution:

$$x_p(t) = \frac{t}{4} \sin(2t).$$

Thus, the general solution is:

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{t}{4} \sin(2t).$$

The table below outlines typical guesses for $x_p(t)$ based on the form of $f(t)$:

$f(t)$	Guess for $x_p(t)$
$P_n(t)$	$At^n + Bt^{n-1} + \cdots$
$e^{\alpha t}$	$Ae^{\alpha t}$
$\sin(\beta t), \cos(\beta t)$	$A \sin(\beta t) + B \cos(\beta t)$
$P_n(t)e^{\alpha t}$	$(At^n + Bt^{n-1} + \cdots)e^{\alpha t}$
$P_n(t) \sin(\beta t), P_n(t) \cos(\beta t)$	$(At^n + Bt^{n-1} + \cdots) \sin(\beta t)$ or $\cos(\beta t)$
$e^{\alpha t} \sin(\beta t), e^{\alpha t} \cos(\beta t)$	$Ae^{\alpha t} \sin(\beta t) + Be^{\alpha t} \cos(\beta t)$

Note that if the guess involves terms that are already part of $x_h(t)$, we must multiply by t to avoid redundancy.

5.3 Resonance in Forced Oscillations

Resonance occurs when the frequency of the external forcing function $f(t)$ matches the natural frequency of the system. In such cases, the amplitude of the oscillation grows indefinitely. Resonance is a critical phenomenon in engineering and physics, where it can lead to large-amplitude vibrations and structural failure.

Consider the equation for a forced harmonic oscillator:

$$x'' + \omega_0^2 x = F_0 \cos(\omega t).$$

The general solution consists of:

- The homogeneous solution, which represents free oscillations with natural frequency ω_0.

- The particular solution, which represents the response to the external force.

When $\omega = \omega_0$, the forcing term matches the natural frequency, leading to resonance. The particular solution in this case grows linearly with time.

Example

Solve the equation:

$$x'' + 4x = 2\cos(2t).$$

The natural frequency is $\omega_0 = 2$, and the forcing term also has frequency 2. Thus, resonance occurs. To find the particular solution, we guess:

$$x_p(t) = At\sin(2t) + Bt\cos(2t).$$

Substituting into the equation and solving for A and B, we get:

$$x_p(t) = \frac{t}{2}\sin(2t).$$

Thus, the general solution is:

$$x(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{t}{2}\sin(2t).$$

This solution grows without bound as t increases, indicating resonance.

5.4 Variation of Parameters for Second-Order Equations

The variation of parameters method is a technique used to find a particular solution to a nonhomogeneous second-order linear differential equation. The general form of such an equation is:

$$a_2u''(t) + a_1u'(t) + a_0u(t) = g(t)$$

where a_2, a_1, a_0 are constants, and $g(t)$ is a given function.

The method involves using the general solution of the corresponding homogeneous equation:

$$a_2u''(t) + a_1u'(t) + a_0u(t) = 0$$

Let the general solution to the homogeneous equation be:

$$u_h(t) = C_1u_1(t) + C_2u_2(t)$$

where $u_1(t)$ and $u_2(t)$ are linearly independent solutions to the homogeneous equation.

For the nonhomogeneous equation, we assume a particular solution of the form:

$$u_p(t) = v_1(t)u_1(t) + v_2(t)u_2(t)$$

where $v_1(t)$ and $v_2(t)$ are functions to be determined.

To find $v_1(t)$ and $v_2(t)$, we impose the condition:

$$v_1'(t)u_1(t) + v_2'(t)u_2(t) = 0$$

and differentiate the assumed solution:

$$u_p'(t) = v_1'(t)u_1(t) + v_2'(t)u_2(t)$$

Substituting this into the original equation and solving for $v_1'(t)$ and $v_2'(t)$, we get the system of equations:

$$\begin{aligned} v_1'(t)u_1(t) + v_2'(t)u_2(t) &= 0 \\ v_1'(t)u_1'(t) + v_2'(t)u_2'(t) &= g(t) \end{aligned}$$

Solving this system for $v_1'(t)$ and $v_2'(t)$, we can integrate to find $v_1(t)$ and $v_2(t)$, and thus obtain the particular solution $u_p(t)$.

Example

Solve the nonhomogeneous differential equation:

$$u'' + u = \tan t$$

The homogeneous equation $u'' + u = 0$ has the general solution:

$$u_h(t) = C_1 \cos t + C_2 \sin t$$

Using variation of parameters, we assume the particular solution is of the form:

$$u_p(t) = v_1(t) \cos t + v_2(t) \sin t$$

Substituting into the equations for $v_1'(t)$ and $v_2'(t)$, we solve for $v_1(t)$ and $v_2(t)$, and find the particular solution. Finally, the general solution is:

$$u(t) = C_1 \cos t + C_2 \sin t + u_p(t)$$

Lecture 6: Higher-Order Differential Equations and Systems of Differential Equations

Higher-order differential equations involve derivatives of order higher than two. These equations often appear in advanced applications in physics and engineering.

Definition

A *higher-order differential equation* is an equation that involves the unknown function $u(t)$ and its derivatives up to some order n. The general form is:

$$F(t, u, u', u'', \ldots, u^{(n)}) = 0$$

6.1 Linear Higher-Order Differential Equations

A linear higher-order differential equation has the form:

$$a_n(t)u^{(n)} + a_{n-1}(t)u^{(n-1)} + \cdots + a_1(t)u' + a_0(t)u = g(t)$$

where $a_0(t), a_1(t), \ldots, a_n(t)$ and $g(t)$ are given functions.

Example

Consider the third-order differential equation:

$$u''' - 6u'' + 11u' - 6u = 0$$

To solve this equation, we assume a solution of the form $u(t) = e^{rt}$. Substituting this into the equation gives the characteristic equation:

$$r^3 - 6r^2 + 11r - 6 = 0$$

Solving the characteristic equation, we find the roots $r = 1, 2, 3$. Thus, the general solution is:

$$u(t) = C_1e^t + C_2e^{2t} + C_3e^{3t}$$

where C_1, C_2, and C_3 are arbitrary constants.

6.2 Reduction of Order

The reduction of order method is used to find a second, linearly independent solution to a second-order linear differential equation when one solution is already known. This method is particularly useful for solving homogeneous differential equations of the form:

$$a_2u''(t) + a_1u'(t) + a_0u(t) = 0$$

where a_2, a_1, and a_0 are constants or functions of t, and $u(t)$ is the unknown function.

If one solution, say $u_1(t)$, is already known, we can use the reduction of order method to find a second, linearly independent solution $u_2(t)$. The idea is to assume that the second solution has the form:

$$u_2(t) = v(t)u_1(t)$$

where $v(t)$ is a function to be determined. By substituting this expression into the original equation, the problem reduces to a first-order differential equation for $v(t)$, which is easier to solve.

Example

Consider the second-order differential equation:

$$u'' - u' - 6u = 0$$

We are given that $u_1(t) = e^{3t}$ is one solution to the equation. We will now use the reduction of order method to find a second linearly independent solution.

First, we assume that the second solution has the form:

$$u_2(t) = v(t)e^{3t}$$

where $v(t)$ is an unknown function. Next, we compute the derivatives of $u_2(t)$:

$$u_2'(t) = v'(t)e^{3t} + 3v(t)e^{3t}$$

$$u_2''(t) = v''(t)e^{3t} + 6v'(t)e^{3t} + 9v(t)e^{3t}$$

Substituting these expressions into the original differential equation:

$$u_2'' - u_2' - 6u_2 = 0$$

we get:

$$\left(v''(t)e^{3t} + 6v'(t)e^{3t} + 9v(t)e^{3t}\right) - \left(v'(t)e^{3t} + 3v(t)e^{3t}\right) - 6v(t)e^{3t} = 0$$

Simplifying the equation:

$$v''(t)e^{3t} + 6v'(t)e^{3t} + 9v(t)e^{3t} - v'(t)e^{3t} - 3v(t)e^{3t} - 6v(t)e^{3t} = 0$$

$$v''(t)e^{3t} + 5v'(t)e^{3t} = 0$$

Since $e^{3t} \neq 0$, we can divide by e^{3t}, which simplifies the equation to:

$$v''(t) + 5v'(t) = 0$$

This is now a first-order equation for $v'(t)$. Let $w(t) = v'(t)$, then the equation becomes:

$$w'(t) + 5w(t) = 0$$

This is a separable differential equation. We can solve it by separating the variables:

$$\frac{dw}{w} = -5dt$$

Integrating both sides, we obtain:

$$\ln|w| = -5t + C_1$$

Exponentiating both sides gives:

$$w(t) = C_1 e^{-5t}$$

Since $w(t) = v'(t)$, we now integrate $v'(t)$ to find $v(t)$:

$$v(t) = \int C_1 e^{-5t}\, dt = \frac{C_1}{-5} e^{-5t} + C_2$$

Now, we substitute $v(t)$ back into the expression for $u_2(t)$:

$$u_2(t) = v(t)e^{3t} = \left(\frac{C_1}{-5}e^{-5t} + C_2\right)e^{3t}$$

Simplifying, we get:

$$u_2(t) = \frac{C_1}{-5}e^{-2t} + C_2e^{3t}$$

Since C_2e^{3t} is a multiple of the first solution $u_1(t) = e^{3t}$, we discard this term, and the second linearly independent solution is:

$$u_2(t) = e^{-2t}$$

Thus, the general solution to the differential equation is:

$$u(t) = C_1e^{3t} + C_2e^{-2t}$$

The reduction of order method allows us to find a second, linearly independent solution by assuming a solution of the form $u_2(t) = v(t)u_1(t)$, where $u_1(t)$ is the known solution. This approach transforms the original second-order differential equation into a first-order equation for $v(t)$, which is simpler to solve.

6.3 Cauchy-Euler Equations

The Cauchy-Euler equation is a special type of linear differential equation with variable coefficients. It has the general form:

$$t^n u^{(n)}(t) + a_{n-1}t^{n-1}u^{(n-1)}(t) + \cdots + a_0u(t) = 0$$

The solution method is based on assuming a solution of the form $u(t) = t^r$, where r is a constant to be determined.

Example

Consider the second-order Cauchy-Euler equation:

$$t^2u'' - 3tu' + 3u = 0$$

Substituting $u(t) = t^r$, we get the characteristic equation:

$$r(r-1) - 3r + 3 = 0$$

Simplifying, we find:

$$r^2 - 4r + 3 = 0$$

The roots are $r = 1$ and $r = 3$. Therefore, the general solution is:

$$u(t) = C_1t + C_2t^3$$

where C_1 and C_2 are arbitrary constants.

For nonhomogeneous Cauchy-Euler equations, methods such as variation of parameters or undetermined coefficients can be used to find particular solutions.

Lecture 7: Linear Systems of Differential Equations

Linear systems of differential equations are widely used in various fields such as physics, biology, and engineering. These systems involve multiple interdependent variables, and the behavior of such systems can be modeled using matrices and vectors. The general form of a linear system of first-order differential equations is:

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where $\mathbf{x}(t)$ is a vector of unknown functions, and A is a constant matrix.

Consider a system of two linear differential equations:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2, \\ x_2' &= a_{21}x_1 + a_{22}x_2. \end{aligned}$$

This system can be written more compactly in matrix form:

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \text{where} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

The solution to this system depends on the eigenvalues and eigenvectors of the matrix A.

7.1 Eigenvalue and Eigenvector Method

The general solution of a system of linear differential equations can be found by solving the eigenvalue problem for the matrix A. If λ is an eigenvalue of A and $\mathbf{v}$ is the corresponding eigenvector, the general solution is given by:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2,$$

where λ_1, λ_2 are the eigenvalues of A, and $\mathbf{v}_1, \mathbf{v}_2$ are the corresponding eigenvectors.

Example

Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix} \mathbf{x}.$$

To solve this system, we first find the eigenvalues of the matrix A. The characteristic equation is:

$$\det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = (4-\lambda)(3-\lambda) = 0.$$

Thus, the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 3$. The eigenvectors are found by solving:

$$(A - 4I)\mathbf{v}_1 = 0 \quad \text{and} \quad (A - 3I)\mathbf{v}_2 = 0.$$

This yields the eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore, the general solution is:

$$\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For systems with real and distinct eigenvalues, the general solution is straightforward, as in the previous example. However, when the eigenvalues are complex, the solution takes a different form. Suppose $\lambda = \alpha + i\beta$ is a complex eigenvalue with the corresponding eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$. The solution can be written as:

$$\mathbf{x}(t) = e^{\alpha t} \left(c_1 \mathbf{u} \cos(\beta t) - c_2 \mathbf{w} \sin(\beta t) \right),$$

where $\mathbf{u}$ and $\mathbf{w}$ are the real and imaginary parts of the eigenvector, respectively.

Example

Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

The characteristic equation is:

$$\det(A - \lambda I) = \lambda^2 + 1 = 0,$$

so the eigenvalues are $\lambda = \pm i$. The general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

This represents oscillatory motion, as expected for systems with purely imaginary eigenvalues.

7.2 Phase Plane Analysis

The behavior of solutions to a system of linear differential equations can be understood geometrically by plotting the solution trajectories in the phase plane. The type of behavior observed depends on the nature of the eigenvalues:

- **Real, distinct eigenvalues**: The trajectories are straight lines or hyperbolas, corresponding to nodes or saddle points.
- **Complex eigenvalues**: The trajectories are spirals or circles, corresponding to foci or centers.
- **Repeated eigenvalues**: The trajectories are straight lines or degenerate nodes.

Example

For the system:

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x},$$

the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$. The phase plane shows a saddle point, with trajectories moving away from the origin along the x_1-axis and toward the origin along the x_2-axis.

7.3 Nonhomogeneous Systems

A nonhomogeneous system of the form:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t),$$

can be solved by finding the general solution to the homogeneous system and adding a particular solution to the nonhomogeneous system:

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t).$$

The particular solution $\mathbf{x}_p(t)$ can be found using methods such as undetermined coefficients or variation of parameters.

Example

Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

First, solve the homogeneous system $\mathbf{x}' = A\mathbf{x}$. The eigenvalues of A are found by solving the characteristic equation:

$$\det(A - \lambda I) = \lambda^2 - 5\lambda - 2 = 0,$$

yielding the eigenvalues $\lambda_1 = \frac{5+\sqrt{33}}{2}$ and $\lambda_2 = \frac{5-\sqrt{33}}{2}$. After finding the general solution for the homogeneous system, we apply undetermined coefficients to find the particular solution for the non-homogeneous system.

7.4 Compartmental Model: Pesticide Transfer

Consider a compartmental model where a pesticide is sprayed on the soil at time $t = 0$. The pesticide transfers between the crops and soil, with degradation happening in the soil. Let $x(t)$ represent the concentration of the pesticide in the crops, and $y(t)$ represent the concentration in the soil. The differential equations governing this system are:

$$\frac{dx}{dt} = -\beta x + \alpha y,$$
$$\frac{dy}{dt} = \beta x - (\alpha + \gamma) y,$$

where:

- α is the rate of pesticide uptake by the crops,
- β is the rate of transfer from crops back to the soil,
- γ is the rate of natural degradation of pesticide in the soil.

This system can be written in matrix form as:

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \text{where} \quad A = \begin{pmatrix} -\beta & \alpha \\ \beta & -(\alpha + \gamma) \end{pmatrix}, \quad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Solving this system involves finding the eigenvalues and eigenvectors of matrix A, which describes the transfer dynamics between the compartments of crops and soil.

Definition

A *vector field* is a function that assigns a vector to every point in a subset of space. Formally, if $\mathcal{D} \subseteq \mathbb{R}^n$ is a domain in $\mathbb{R}^n$, then a vector field is a function $\mathbf{F} : \mathcal{D} \to \mathbb{R}^n$ such that:

$$\mathbf{F}(x_1, x_2, \ldots, x_n) = \begin{pmatrix} F_1(x_1, x_2, \ldots, x_n) \\ F_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ F_n(x_1, x_2, \ldots, x_n) \end{pmatrix}$$

where each component $F_i(x_1, x_2, \ldots, x_n)$ is a scalar-valued function, and $\mathbf{F}(x_1, x_2, \ldots, x_n)$ is a vector at each point $(x_1, x_2, \ldots, x_n) \in \mathcal{D}$.
For a two-dimensional vector field in $\mathbb{R}^2$, the function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ can be written as:

$$\mathbf{F}(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix}$$

where $F_1(x, y)$ and $F_2(x, y)$ are scalar functions representing the components of the vector field at any point $(x, y) \in \mathbb{R}^2$.

Example

Consider the system of differential equations representing simple harmonic motion:

$$x' = y, \quad y' = -4x,$$

where $x(t)$ represents the displacement and $y(t)$ is the velocity of an oscillating object. The vector field at the point (x, y) is given by:

$$\mathbf{F} = (y, -4x).$$

This system describes oscillatory motion. By eliminating t, we can show that the orbits are ellipses in the xy-plane. Solving these equations yields the parametric solutions:

$$x(t) = \cos(2t), \quad y(t) = -2\sin(2t),$$

or equivalently:

$$x^2 + \frac{y^2}{4} = 1,$$

which describes an ellipse. The solution represents harmonic oscillations around the origin, with a period of π. The behavior of the system can be understood geometrically by analyzing the phase plane, where the trajectories are closed orbits around the origin, consistent with the expected motion of harmonic oscillators.

Lecture 8: Introduction to Nonlinear Systems

In the study of differential equations, systems are classified as *linear* or *nonlinear* based on whether the equations involve linear or nonlinear functions of the unknown variables and their derivatives. A **linear system** has the general form:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b}(t), \tag{8.1}$$

where $\mathbf{x}$ is a vector of state variables, A is a matrix of constants, and $\mathbf{b}(t)$ is a vector of functions linear in t. Linear systems are well-understood, and powerful analytic methods exist for solving them.

In contrast, a **nonlinear system** involves nonlinear functions of the variables:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \tag{8.2}$$

where $\mathbf{f}$ is a nonlinear function. Nonlinear systems are far more complex, and their solutions often exhibit rich and varied behaviors not present in linear systems, such as multiple equilibria, limit cycles, chaos, and bifurcations.

Most nonlinear differential equations cannot be solved analytically. This lack of general solvability arises because nonlinear terms can create intricate interactions between variables, leading to complex dynamics that resist closed-form solutions. For example, consider the nonlinear equation:

$$\frac{dx}{dt} = x^2. \tag{8.3}$$

While this particular equation can be solved analytically, adding more variables or nonlinear interactions typically complicates matters beyond analytic reach.

Given the difficulty in finding exact solutions, qualitative and geometric methods become essential tools in understanding nonlinear systems. Techniques such as phase plane analysis, linearization, and numerical simulations allow us to investigate the behavior of solutions near equilibrium points, the stability of systems, and the possible long-term dynamics.

8.1 Equilibrium Points and Stability

An **equilibrium point** (or critical point, stationary point) of a dynamical system is a point $\mathbf{x}_0$ in the phase space where the system's derivatives vanish:

$$\left.\frac{d\mathbf{x}}{dt}\right|_{\mathbf{x}=\mathbf{x}_0} = \mathbf{0}. \tag{8.4}$$

At an equilibrium point, the state of the system does not change over time. Finding equilibrium points involves solving the system of equations:

$$\mathbf{f}(\mathbf{x}_0) = \mathbf{0}. \tag{8.5}$$

The stability of an equilibrium point refers to how the system behaves when it is slightly perturbed from that point. An equilibrium point can be:

- **Stable**: Solutions starting near the equilibrium remain close to it for all future time.
- **Asymptotically Stable**: Solutions not only remain close but also tend to the equilibrium as $t \to \infty$.
- **Unstable**: Solutions diverge from the equilibrium point when slightly perturbed.

Example Consider the nonlinear ordinary differential equation:

$$\frac{dx}{dt} = x(1-x). \tag{8.6}$$

To find the equilibrium points, set $\dfrac{dx}{dt} = 0$:

$$\begin{aligned} &x(1-x) = 0 \\ \Rightarrow \quad &x = 0 \quad \text{or} \quad x = 1. \end{aligned}$$

Thus, the system has equilibrium points at $x = 0$ and $x = 1$.
To determine the stability of these points, examine the sign of $\dfrac{dx}{dt}$ near each equilibrium:

- **For** $x < 0$:

  Since $x < 0$ and $1 - x > 1$, we have:

  $$x(1-x) < 0 \implies \frac{dx}{dt} < 0.$$

  Therefore, x decreases.

- **For** $0 < x < 1$:

  Both $x > 0$ and $1 - x > 0$, so:

  $$x(1-x) > 0 \implies \frac{dx}{dt} > 0.$$

  Thus, x increases.

- **For** $x > 1$:

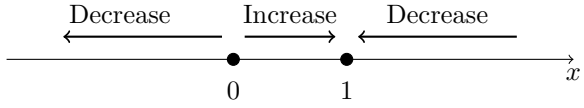
  Here, $x > 1$ and $1 - x < 0$, which gives:

  $$x(1-x) < 0 \implies \frac{dx}{dt} < 0.$$

  So, x decreases.

Thus,

- At $x = 0$, any small positive perturbation causes x to increase away from 0, making $x = 0$ an **unstable** equilibrium point.
- At $x = 1$, small deviations result in x returning to 1, so $x = 1$ is a **stable** equilibrium point.



8.2 Linearization Technique

Linearization is a technique for analyzing the local behavior of nonlinear dynamical systems near equilibrium points by approximating the nonlinear system with a linear one in the vicinity of the equilibrium. For a general nonlinear system of differential equations,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{x})$ is a continuously differentiable vector-valued function, an equilibrium point $\mathbf{x}_0$ is defined by $\mathbf{f}(\mathbf{x}_0) = 0$.

To approximate the system near $\mathbf{x}_0$, we expand $\mathbf{f}(\mathbf{x})$ in a Taylor series around $\mathbf{x}_0$:

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \text{higher-order terms}.$$

Since $\mathbf{f}(\mathbf{x}_0) = 0$, this simplifies to

$$\mathbf{f}(\mathbf{x}) \approx D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0),$$

where $D\mathbf{f}(\mathbf{x}_0)$ is the Jacobian matrix evaluated at the equilibrium point $\mathbf{x}_0$.

Near $\mathbf{x}_0$, higher-order terms in the Taylor expansion become negligible compared to the linear terms. Thus, we approximate the system by the linearized equation

$$\frac{d\mathbf{x}}{dt} \approx D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0),$$

where the matrix $D\mathbf{f}(\mathbf{x}_0)$ governs the dynamics of small perturbations near $\mathbf{x}_0$.

The Jacobian matrix $D\mathbf{f}(\mathbf{x}_0)$ for a system with n equations and n variables $\{x_1, x_2, \ldots, x_n\}$ is given by

$$D\mathbf{f}(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_0}.$$

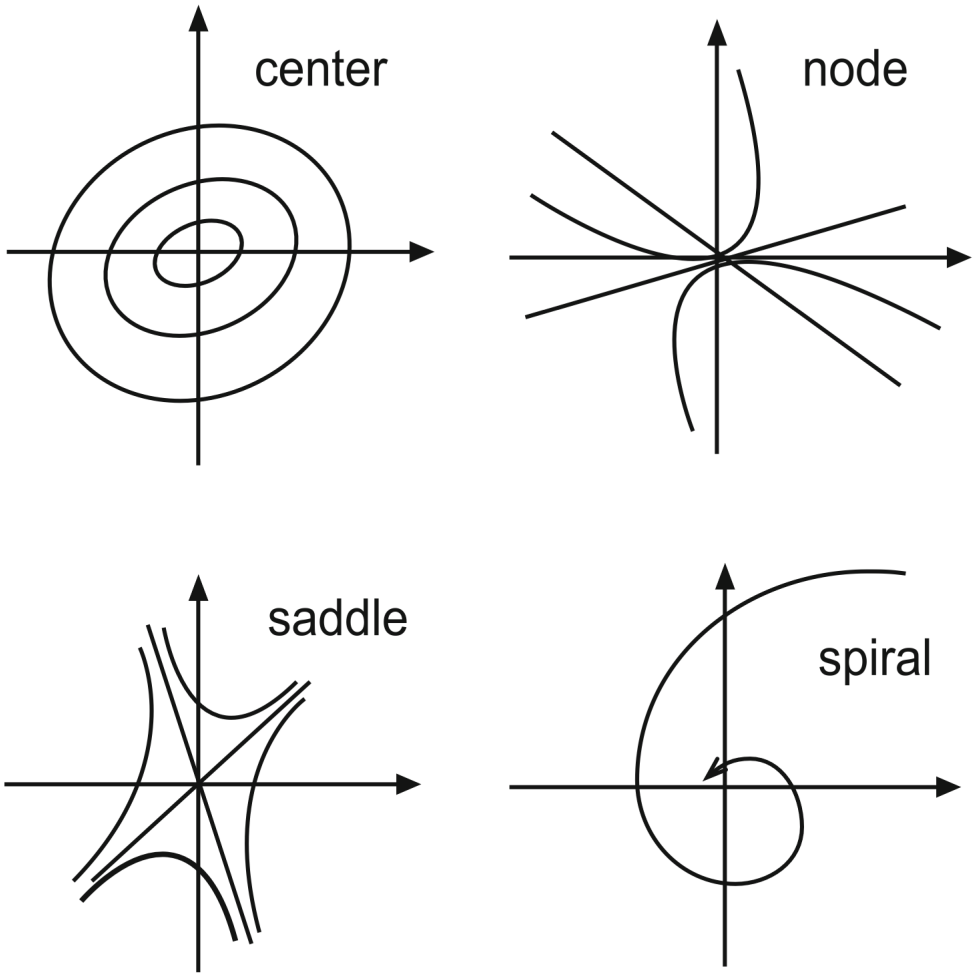
The stability of the equilibrium point $\mathbf{x}_0$ depends on the eigenvalues of the Jacobian matrix $D\mathbf{f}(\mathbf{x}_0)$:

- If all eigenvalues have negative real parts, the equilibrium is asymptotically stable.
- If any eigenvalue has a positive real part, the equilibrium is unstable.
- If the eigenvalues are purely imaginary, the system may exhibit neutral stability, requiring further analysis.

Procedure for Linearization Analysis

To perform a linearization analysis, the following steps are useful:

1. Identify the equilibrium points by solving $\mathbf{f}(\mathbf{x}) = 0$.
2. Compute the Jacobian matrix $D\mathbf{f}(\mathbf{x}_0)$ at each equilibrium.
3. Calculate the eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$.
4. Classify the stability and behavior of the system near the equilibrium based on the eigenvalues.



Types of Equilibrium Points

Equilibrium points in dynamical systems can be classified based on the eigenvalues of the linearized system around those points. Understanding these classifications helps predict the behavior of trajectories near the equilibrium points.

- **Node**:

  Nodes occur when the eigenvalues are real, distinct, and have the same sign. If both eigenvalues are negative, the node is **stable** (an **attracting node**); trajectories approach the equilibrium point as time progresses. If both eigenvalues are positive, the node is **unstable** (a **repelling node**); trajectories move away from the equilibrium point.

- **Saddle Point**:

  Saddle points occur when the eigenvalues are real and have opposite signs. One eigenvalue is positive, and the other is negative. Trajectories approach the equilibrium point along the stable manifold (associated with the negative eigenvalue) and move away along the unstable manifold (associated with the positive eigenvalue). Saddle points are always **unstable**.

- **Spiral (Focus)**:

Spirals occur when the eigenvalues are complex with nonzero real parts. The real part determines stability: if the real part is negative, trajectories spiral into the equilibrium point (**stable focus**); if positive, they spiral out (**unstable focus**).

- **Center**:

  Centers occur when the eigenvalues are purely imaginary (zero real part). Trajectories around a center are closed orbits; they neither approach nor recede from the equilibrium point. The equilibrium is **neutrally stable**.

The behavior of the system near the equilibrium point $\mathbf{x}_0$ can be classified as follows, depending on the eigenvalues of the Jacobian matrix:

- **Node**: If the eigenvalues are real and of the same sign, trajectories either converge (if negative) or diverge (if positive) directly towards or away from the equilibrium along straight paths.
- **Saddle**: If the eigenvalues have opposite signs, the equilibrium is a saddle point, with trajectories repelled along directions associated with positive eigenvalues and attracted along those associated with negative eigenvalues.
- **Spiral**: If the eigenvalues are complex with non-zero real parts, trajectories spiral into or out of the equilibrium.
- **Center**: If the eigenvalues are purely imaginary, trajectories form closed orbits around the equilibrium, indicating neutral stability.

Example 1 Consider the system

$$x' = -x + xy, \quad y' = -4y + 8xy.$$

To find critical points, we solve $-x + xy = 0$ and $-4y + 8xy = 0$. We find two equilibria: $(0, 0)$ and $\left(\frac{1}{2}, 1\right)$.

The Jacobian $J(x, y)$ is

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -1 + y & x \\ 8y & -4 + 8x \end{bmatrix}.$$

Evaluating at $(0, 0)$, we have

$$J(0, 0) = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix},$$

indicating that $(0, 0)$ is an assintotically stable node. At $\left(\frac{1}{2}, 1\right)$, we find

$$J\left(\frac{1}{2}, 1\right) = \begin{bmatrix} 0 & \frac{1}{2} \\ 8 & 0 \end{bmatrix}.$$

The eigenvalues are ± 2, so $\left(\frac{1}{2}, 1\right)$ is a unstable saddle point.

Example 2 Consider the nonlinear system

$$x' = x - x^3, \quad y' = 2y.$$

The Jacobian $J(x, y)$ is

$$J(x, y) = \begin{bmatrix} 1 - 3x^2 & 0 \\ 0 & 2 \end{bmatrix}.$$

At $(0, 0)$, the eigenvalues are 1 and 2, indicating an unstable node. At $(1, 0)$ and $(-1, 0)$, we find eigenvalues -2 and 2, identifying these points as saddle points. This system demonstrates how eigenvalues can determine node and saddle behaviors based on their signs

Example 3 For the system

$$x' = y^2, \quad y' = -\frac{2}{3}x,$$

the Jacobian at $(0,0)$ is

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ -\frac{2}{3} & 0 \end{bmatrix},$$

which has eigenvalues of zero, meaning the linearization does not apply here. We use an alternative approach to solve for orbits, obtaining $y^3 = -x^2 + C$, which reveals non-standard behaviour with non-linear closed.

Example 4 Consider the system:

$$\frac{dx}{dt} = x(1-y), \tag{8.7}$$

$$\frac{dy}{dt} = y(x-1). \tag{8.8}$$

Equilibrium Points Setting $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$, we find equilibrium points at $(0,0)$, $(1,1)$, $(0,1)$, and $(1,0)$.

Computing the Jacobian The Jacobian matrix is:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-y & -x \\ y & x-1 \end{bmatrix}. \tag{8.9}$$

Evaluating at $(1,1)$ At $(x_0, y_0) = (1,1)$:

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.10}$$

Eigenvalues The characteristic equation is:

$$\det(J - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0. \tag{8.11}$$

Solving $\lambda^2 + 1 = 0$ gives $\lambda = \pm i$.

Since the eigenvalues are purely imaginary, $(1,1)$ is a **center**, indicating neutral stability. However, due to nonlinear terms, the actual stability may differ, and further analysis is required.

Example

The damped pendulum equation is:

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0^2 \sin\theta = 0, \tag{8.12}$$

where:

- θ is the angular displacement.
- γ is the damping coefficient.
- $\omega_0 = \sqrt{\frac{g}{L}}$ is the natural frequency.

Introduce variables:

$$x = \theta, \tag{8.13}$$
$$y = \frac{d\theta}{dt}. \tag{8.14}$$

Then, the system becomes:

$$\frac{dx}{dt} = y, \tag{8.15}$$
$$\frac{dy}{dt} = -\gamma y - \omega_0^2 \sin x. \tag{8.16}$$

Plotting y versus x, we can analyze the trajectories:

- At high damping (γ large), the pendulum returns slowly to equilibrium without oscillating.
- At low damping, the pendulum oscillates with decreasing amplitude.
- Energy dissipation causes spiraling towards the origin (stable focus).

8.3 The Lotka–Volterra Model

The Lotka–Volterra model is a fundamental mathematical model describing predator-prey interactions in ecological systems. Let $x = x(t)$ represent the prey population and $y = y(t)$ represent the predator population. Examples of such interactions include rabbits and foxes, or food fish and sharks.

Model Formulation

In the absence of predators, we assume that the prey population grows exponentially:

$$x' = rx,$$

where r is the intrinsic growth rate of the prey. Similarly, in the absence of prey, the predator population decays at a rate proportional to the population:

$$y' = -my,$$

where m is the per capita mortality rate of the predators.

When interactions between prey and predators occur, they impact both populations. We assume that the rate at which predators consume prey is proportional to the prey population x, so the rate of prey

consumption per predator is ax, where $a > 0$ is a constant known as the predator's functional response. For y predators, the total rate of prey consumption becomes axy, reducing the prey population by this term.

The increase in the predator population depends on prey consumption, with a rate ϵaxy, where ϵ is the conversion efficiency, representing how effectively prey consumption translates to predator reproduction. This yields the following system, known as the Lotka–Volterra equations:

$$x' = rx - axy,$$
$$y' = -my + \epsilon axy.$$

The Lotka–Volterra model has been used to explain oscillatory population dynamics. At equilibrium, both populations do not change over time, so setting $x' = 0$ and $y' = 0$ gives:

$$rx - axy = 0,$$
$$-my + \epsilon axy = 0.$$

These equations have two equilibrium points: $(x, y) = (0, 0)$, representing extinction, and $\left(\frac{m}{\epsilon a}, \frac{r}{a}\right)$, representing coexistence of the prey and predator populations.

Stability of the Equilibrium To analyze the stability of these equilibria, we examine the Jacobian matrix evaluated at each equilibrium. The Jacobian of the system is:

$$J(x, y) = \begin{bmatrix} r - ay & -ax \\ \epsilon ay & -m + \epsilon ax \end{bmatrix}.$$

At $(x, y) = \left(\frac{m}{\epsilon a}, \frac{r}{a}\right)$, substituting these values into the Jacobian gives insights into the nature of the population dynamics near the coexistence equilibrium.

Biological Significance

The Lotka–Volterra model captures cyclic population dynamics, where an increase in prey population leads to an increase in predator population, which eventually reduces the prey population, thus reducing the predator population, allowing the prey population to recover. This cycle is a core concept in population ecology and provides foundational insights into predator-prey dynamics.

To analyze the dynamics of the Lotka–Volterra system, we plot the nullclines in the xy-plane. Nullclines are the curves where either x' or y' is zero, which divides the phase plane into regions of different directional flows.

Nullclines The nullclines of the system are given by:

$$x' = 0 \implies y = \frac{r}{a}, \quad \text{or } x = 0,$$
$$y' = 0 \implies x = \frac{m}{\epsilon a}, \quad \text{or } y = 0.$$

The nullclines $x = 0$ and $y = 0$ represent extinction of one species, while $y = \frac{r}{a}$ and $x = \frac{m}{\epsilon a}$ correspond to steady-state population levels for the prey and predator, respectively.

Direction of the Vector Field The phase plane can be divided into regions based on the signs of x' and y'. For example:

- Above the nullcline $y = \frac{r}{a}$, $x' < 0$ since the prey population is decreasing due to over-predation.
- Below the nullcline $y = \frac{r}{a}$, $x' > 0$ since the prey population grows in the absence of sufficient predators.
- To the right of the nullcline $x = \frac{m}{\epsilon a}$, $y' > 0$ since an abundance of prey supports predator population growth.
- To the left of the nullcline $x = \frac{m}{\epsilon a}$, $y' < 0$ as predators die off in the absence of prey.

By plotting these nullclines and analyzing the vector field, we can construct a phase portrait showing the trajectories of the system.

Stability of Equilibria

Origin $(0,0)$ At the extinction equilibrium $(x,y)=(0,0)$, the Jacobian matrix is:

$$J(0,0)=\begin{bmatrix} r & 0 \\ 0 & -m \end{bmatrix}.$$

The eigenvalues are $r>0$ and $-m<0$, indicating a saddle point. This confirms that the origin is an unstable equilibrium, as small perturbations lead to growth in prey population or decay in predator population.

Coexistence Equilibrium $\left(\frac{m}{\epsilon a},\frac{r}{a}\right)$ At the coexistence equilibrium, substituting $x=\frac{m}{\epsilon a}$ and $y=\frac{r}{a}$ into the Jacobian matrix gives:

$$J\left(\frac{m}{\epsilon a},\frac{r}{a}\right)=\begin{bmatrix} 0 & -\frac{am}{\epsilon a} \\ \frac{\epsilon a r}{a} & 0 \end{bmatrix}=\begin{bmatrix} 0 & -m \\ r & 0 \end{bmatrix}.$$

The characteristic equation for this matrix is:

$$\lambda^2+rm=0.$$

The eigenvalues are purely imaginary: $\lambda=\pm i\sqrt{rm}$. This indicates that the coexistence equilibrium is a center, leading to periodic orbits around the equilibrium point.

Periodic Orbits and Biological Implications

The system exhibits closed, periodic orbits in the phase plane, reflecting oscillations in predator and prey populations. These oscillations arise due to the interdependence of the two species:

- When prey populations rise, predator populations increase due to abundant food.
- As predator populations grow, they suppress prey populations, leading to a decline in prey numbers.
- With fewer prey, predator populations decline due to lack of food, allowing prey populations to recover.

These dynamics result in sustained oscillations.

Orbits and Analytical Solutions

The equation of the orbits can be derived by dividing x' and y' to eliminate t:

$$\frac{dy}{dx}=\frac{y(-m+\epsilon ax)}{x(r-ay)}.$$

Separating variables and integrating gives:

$$\int \frac{r-ay}{y}\,dy=\int \frac{-m+\epsilon ax}{x}\,dx+C.$$

Rewriting and simplifying yields the implicit equation for the orbits:

$$r\ln y-ay=-m\ln x+\epsilon ax+C,$$

which cannot be solved explicitly. However, the periodic orbits can be visualized numerically or using the relationship:

$$y^r e^{-ay}=Cx^m e^{\epsilon ax}.$$

8.4 Population Ecology

Population ecology models play a critical role in understanding predator-prey interactions and their implications for ecosystem dynamics. The classical Lotka–Volterra predator-prey model assumes that the rate of predation (prey consumed per unit time per predator) is proportional to the prey population density. However, this assumption becomes unrealistic as prey density increases. At high prey densities, a predator's consumption rate must eventually plateau due to physical and behavioral limitations, such as time spent handling prey. Recognizing these limitations, C. Holling introduced functional responses that account for the predator's time budget and provide a more accurate description of predator-prey dynamics.

Holling Type II Functional Response

The Holling Type II response addresses the issue of predation rates reaching a maximum threshold as prey density increases. Holling reasoned that a predator's time budget is divided into searching for prey (T_s) and handling prey (T_h). The total time available to a predator is given by

$$T = T_s + T_h.$$

The number of prey captured by a predator, denoted N, depends on the prey density x and the searching time T_s, such that:

$$N = aT_s x,$$

where a is the encounter rate constant, representing the predator's efficiency in finding prey.

Assuming that the handling time T_h is proportional to the number of prey captured, we write:

$$T_h = hN,$$

where h is the handling time per prey. Substituting $T_s = T - T_h$ into the equation for N, we obtain:

$$N = a(T - hN)x.$$

Solving for the predation rate N/T, we find:

$$\frac{N}{T} = \frac{ax}{1 + ahx}.$$

This expression, known as the Holling Type II functional response, reflects the saturating nature of predation at high prey densities. The predation rate approaches a maximum value of $1/h$ as $x \to \infty$, representing the predator's physical and behavioral constraints. When applied to a population of y predators, the overall predation rate becomes:

$$\frac{axy}{1 + ahx}.$$

Holling Type III Functional Response

The Holling Type III response introduces a density-dependent encounter rate, where the efficiency of predation increases as prey density surpasses a certain threshold. This behavior models phenomena such as prey image formation, where predators become more effective at capturing prey once they learn to recognize it. If the encounter rate a is proportional to prey density ($a = bx$), the predation rate is given by:

$$\frac{N}{T} = \frac{bx^2}{1 + bhx^2}.$$

This sigmoidal response, characteristic of the Holling Type III model, captures the initial inefficiency of predators at low prey densities and their subsequent efficiency as prey density increases. It is particularly useful for modeling predator behaviors involving selective targeting or learning processes.

Incorporating Holling Responses into the Lotka–Volterra Model

Replacing the linear per-predator feeding rate ax in the classical Lotka–Volterra model with the Holling Type II functional response yields the system:

$$x' = rx - \frac{axy}{1+ahx},$$

$$y' = -my + \epsilon\frac{axy}{1+ahx}.$$

Here:

- r: intrinsic growth rate of the prey population,
- m: mortality rate of predators,
- ϵ: efficiency of converting consumed prey into predator growth.

This system introduces saturation effects in predation, providing a more biologically realistic framework compared to the classical model.

Taking this modification further, we replace the exponential prey growth term rx with a logistic growth term $rx\left(1-\frac{x}{K}\right)$, where K represents the prey population's carrying capacity. The resulting model is known as the Rosenzweig–MacArthur model:

$$x' = rx\left(1-\frac{x}{K}\right) - \frac{axy}{1+ahx},$$

$$y' = -my + \epsilon\frac{axy}{1+ahx}.$$

This model balances prey growth limitations and predator-prey interactions, offering a robust description of ecological dynamics.

Models incorporating Holling responses address critical ecological questions, including:

- Under what conditions do predator and prey populations reach stable equilibria, oscillate, or drive each other to extinction?
- How does the carrying capacity K affect population stability and persistence?
- What happens when environmental factors alter predator efficiency or prey growth rates?

For example, the Holling Type II response predicts that increasing prey density beyond a certain point does not proportionally increase predation, preventing over-predation and allowing prey populations to recover. Meanwhile, the Holling Type III response models the predator's adaptive behaviors, emphasizing the importance of prey density thresholds in maintaining ecological balance.

Limitations of Ecological Models

While these models provide valuable insights into predator-prey dynamics, they are simplifications of complex natural systems. Real-world interactions often involve multiple species and are influenced by abiotic factors, such as temperature or habitat structure. Moreover, the parameters a, h, r, and m may vary across ecosystems, making generalized predictions challenging. Despite these limitations, models like the Rosenzweig–MacArthur framework remain essential tools for understanding and managing ecological systems.

8.5 Dimensionless Equations

Mathematical models are often complicated by the presence of several parameters, making analysis of solutions tedious and computationally demanding. Moreover, the impact of individual parameters on the behavior of the system can be difficult to discern. A common approach to simplify such models and improve their analytical and computational tractability is to reformulate them in terms of *dimensionless variables*. This process, known as **non-dimensionalization**, reduces the number of parameters and eliminates dimensions, allowing for a more general and economical analysis.

The process of non-dimensionalization is rooted in the observation that many problems exhibit inherent scaling properties. By identifying natural scales in the problem, one can define new variables that encapsulate relative rather than absolute quantities. For instance:

- Time can be scaled relative to a characteristic timescale of the system.
- Populations can be scaled relative to carrying capacities.
- Rates can be scaled relative to maximum achievable rates.

These transformations often reveal underlying structures and symmetries of the system that are not immediately apparent in the original formulation.

Example. Consider the modified Lotka–Volterra model with logistic prey growth:

$$x' = rx\left(1 - \frac{x}{K}\right) - axy,$$

$$y' = -my + bxy.$$

This model contains five parameters (r, K, a, b, and m), making analysis challenging. By reformulating the model using dimensionless variables, we aim to reduce the number of parameters.

The model involves three main variables:

- x: prey population,
- y: predator population,
- t: time.

The associated dimensions, denoted using square brackets, are:

$$[t] = \text{time}, \quad [x] = \text{prey}, \quad [y] = \text{predators}.$$

The parameters r, K, a, b, and m have the following dimensions:

$$[r] = [m] = \frac{1}{\text{time}}, \quad [K] = \text{prey}, \quad [a] = \frac{1}{\text{time} \cdot \text{predator}}, \quad [b] = \frac{1}{\text{time} \cdot \text{prey}}.$$

To eliminate dimensions, we introduce new variables by dividing each quantity by a characteristic scale with the same dimension. We define:

$$\tau = tr, \quad X = \frac{x}{K}, \quad Y = \frac{y}{\frac{r}{a}}.$$

Here:

- τ scales time relative to the inverse of the prey growth rate $1/r$,
- X scales the prey population relative to its carrying capacity K,
- Y scales the predator population relative to r/a, a measure of predation efficiency.

Using the chain rule, we relate the derivatives of the dimensionless variables to those of the original variables:

$$x' = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}(KX)}{\mathrm{d}(r^{-1}\tau)} = Kr\frac{\mathrm{d}X}{\mathrm{d}\tau},$$

$$y' = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}\left(\frac{r}{a}Y\right)}{\mathrm{d}(r^{-1}\tau)} = \frac{r^2}{a}\frac{\mathrm{d}Y}{\mathrm{d}\tau}.$$

Substituting these expressions into the original equations gives:

$$Kr\frac{\mathrm{d}X}{\mathrm{d}\tau} = rKX(1-X) - aKX\frac{r}{a}Y,$$

$$\frac{r^2}{a}\frac{\mathrm{d}Y}{\mathrm{d}\tau} = -m\frac{r}{a}Y + bKX\frac{r}{a}Y.$$

Simplifying, we obtain:

$$\frac{\mathrm{d}X}{\mathrm{d}\tau} = X(1-X) - XY,$$

$$\frac{\mathrm{d}Y}{\mathrm{d}\tau} = -\sigma Y + \rho XY,$$

where the dimensionless parameters are:

$$\sigma = \frac{m}{r}, \quad \rho = \frac{bK}{r}.$$

The dimensionless system is:

$$\frac{\mathrm{d}X}{\mathrm{d}\tau} = X(1-X) - XY,$$

$$\frac{\mathrm{d}Y}{\mathrm{d}\tau} = -\sigma Y + \rho XY.$$

This system contains only two parameters (σ and ρ), significantly reducing the complexity of the analysis. The parameter σ represents the ratio of predator mortality to prey growth rate, while ρ reflects the product of predator efficiency and prey carrying capacity normalized by the prey growth rate.

Reformulating models in terms of dimensionless variables has several benefits:

- It reduces the number of parameters, simplifying numerical simulations and analytical studies.
- It reveals scaling laws and invariant properties of the system.
- It enables generalization, as results apply across systems with different absolute parameter values but the same dimensionless combinations.

While the process of non-dimensionalization requires careful attention to scaling and parameter relationships, it is a powerful tool for analyzing complex models in biology, engineering, and physics.

8.6 The SIR Model for Epidemics

The SIR model is a compartmental framework that divides a fixed population of size N into three categories:

- $S(t)$: the number of susceptible individuals at time t,
- $I(t)$: the number of infected individuals at time t,
- $R(t)$: the number of removed individuals (recovered or deceased) at time t.

The total population remains constant:

$$N = S(t) + I(t) + R(t).$$

Model Assumptions

The SIR model is based on the following assumptions:

- No births, deaths (other than disease-related), or immigration occur during the model's timeframe.
- The population is well-mixed, meaning all individuals interact with equal probability.
- Recovery confers permanent immunity, and individuals progress through $S \to I \to R$ compartments.

Equations of the Model

The dynamics of the disease spread are described by:

$$S' = -aSI, \quad I' = aSI - rI, \quad R' = rI,$$

where:

- a: the effective transmission coefficient, representing the rate of contact leading to infection,
- r: the recovery rate, where $1/r$ is the average infectious period.

The first equation reflects the reduction in susceptible individuals due to infections. The second describes the change in infected individuals, accounting for new infections and recoveries. The third tracks the accumulation of removed individuals.

Phase Plane Analysis and R_0

In the SI phase plane, the trajectories describe the evolution of S and I over time. Key features include:

- The disease spreads if the number of susceptible individuals S exceeds r/a, as $I' > 0$.
- The basic reproductive number R_0 determines epidemic potential:

$$R_0 = \frac{aS(0)}{r}.$$

If $R_0 > 1$, the disease spreads; if $R_0 < 1$, it dies out.

Lecture 9: Advanced dynamical systems modeling

9.1 SEIR Model Dynamics

The SEIR model is a compartmental framework used extensively in epidemiology to describe the dynamics of infectious diseases within a population. The population is divided into compartments, with individuals transitioning between them over time. The compartments in the basic SEIR model are as follows:

- S: Susceptible individuals who have not been exposed to the pathogen.
- E: Exposed individuals who are infected but not yet infectious (latent period).
- I: Infectious individuals capable of transmitting the pathogen.
- R: Recovered individuals who are no longer susceptible or infectious.

The transitions between these compartments are governed by a system of ordinary differential equations (ODEs), which describe the rate of change of the population in each compartment over time:

$$\dot{S} = -\lambda S, \tag{9.1}$$
$$\dot{E} = \lambda S - \gamma_E E, \tag{9.2}$$
$$\dot{I} = \gamma_E E - \gamma_I I, \tag{9.3}$$
$$\dot{R} = \gamma_I I. \tag{9.4}$$

Here, λ is the force of infection, γ_E is the rate at which exposed individuals become infectious, and γ_I is the recovery rate.

The force of infection λ captures the rate at which susceptible individuals become infected. It depends on the infectious population and the contact structure:

$$\lambda = \beta \frac{I}{N},$$

where β is the transmission rate, and $N = S + E + I + R$ is the total population size.

Extensions for Age Structure and Epidemiological Detail

To improve realism, the SEIR model can be extended to account for age-structured populations and additional epidemiological details. In such models, the population is divided into n age groups, and the compartments S, E, I, R are further indexed by age class i:

$$\dot{S}_i = -\lambda_i S_i, \tag{9.5}$$
$$\dot{E}_i = \lambda_i S_i - \gamma_E E_i, \tag{9.6}$$
$$\dot{I}_i = \gamma_E E_i - \gamma_I I_i, \tag{9.7}$$
$$\dot{R}_i = \gamma_I I_i. \tag{9.8}$$

The force of infection λ_i for age group i now depends on interactions across all age groups:

$$\lambda_i = \sum_j M_{ij} \frac{\beta_j I_j}{N_j},$$

where M_{ij} is the contact matrix specifying the average number of contacts between individuals in age groups i and j.

Incorporating Non-Pharmaceutical Interventions (NPIs)

Non-pharmaceutical interventions (NPIs), such as lockdowns and social distancing, are incorporated into the model by modulating the transmission rate β over time. The transmission rate is parameterized as a time-varying function $\beta(t)$, which reflects changes in contact behavior and other factors, including government-imposed restrictions:

$$\beta(t) = b(t) \cdot \beta_0,$$

where $b(t)$ is a scaling function. In the original model under review, $b(t)$ was represented as a piecewise linear function with breakpoints aligned with intervention dates. However, this representation imposes restrictive assumptions. To address this, $b(t)$ can be modeled using adaptive smoothing splines:

$$b(t) = \sum_{k=1}^{K} \theta_k \phi_k(t),$$

where $\phi_k(t)$ are basis functions, and θ_k are coefficients estimated from the data.

9.2 Malaria Transmission Model

Malaria is a vector-borne disease, meaning that its transmission depends on the interaction between human hosts and a vector organism—in this case, the female *Anopheles* mosquito. The disease's infectious agent is a protozoan parasite, which is introduced into the human bloodstream when an infected mosquito bites. The parasite undergoes part of its lifecycle inside the mosquito and is then transmitted back to humans, completing a complex cycle of infection.

Unlike directly transmitted diseases such as measles or influenza, malaria exhibits a criss-cross pattern of transmission:

- Humans transmit the disease to mosquitoes when an infected person is bitten.
- Mosquitoes transmit the disease to humans through subsequent bites.

This interaction requires a mathematical model to capture the dynamics between the host (humans) and the vector (mosquitoes). The foundational model for malaria transmission, developed by R. Ross (1911) and later refined by G. Macdonald (1957), makes several simplifying assumptions:

- Human victims have no immunity, but they recover without dying.
- The mosquito and human populations remain constant.
- The timescale of disease dynamics is much faster than the population dynamics of either species.

Variables and Parameters

Let:

- H_T: total human population,
- M_T: total mosquito population,

- $H(t)$: number of infected humans at time t,
- $M(t)$: number of infected mosquitoes at time t.

The governing equations for the system are derived from the following assumptions:

1. The infection rate of humans depends on the number of infected mosquitoes (M), the biting rate (a), and the fraction of bites that lead to human infection (b):

$$\frac{\mathrm{d}H}{\mathrm{d}t} = abM\frac{H_T - H}{H_T} - rH.$$

   Here, r is the per capita recovery rate of infected humans.

2. The infection rate of mosquitoes depends on the number of infected humans (H), the biting rate (a), and the fraction of bites that lead to mosquito infection (c):

$$\frac{\mathrm{d}M}{\mathrm{d}t} = ac(M_T - M)\frac{H}{H_T} - \mu M.$$

   Here, μ is the per capita death rate of infected mosquitoes.

Dimensionless Variables

To simplify the analysis, the equations are expressed in terms of the fractions of infected humans and mosquitoes:

$$h = \frac{H}{H_T}, \quad m = \frac{M}{M_T}.$$

Substituting these into the governing equations yields:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \alpha m(1-h) - rh, \quad \frac{\mathrm{d}m}{\mathrm{d}t} = \beta h(1-m) - \mu m,$$

where:

$$\alpha = ab\frac{M_T}{H_T}, \quad \beta = ac.$$

Phase Plane Analysis

The system's dynamics can be analyzed geometrically in the h-m phase plane:

- Nullclines are found by setting the right-hand sides of the equations to zero:

$$m = \frac{rh}{\alpha(1-h)} \quad \text{(h nullcline)}, \quad m = \frac{\beta h}{\mu + \beta h} \quad \text{(m nullcline)}.$$

- The h-nullcline is concave up and asymptotes vertically as $h \to 1$, while the m-nullcline is concave down and asymptotes horizontally as $m \to 1$.

Two cases arise:

1. If $\frac{\beta}{\mu} < \frac{r}{\alpha}$, the nullclines intersect only at the origin $(h, m) = (0, 0)$, which is a stable equilibrium. The disease dies out.

2. If $\frac{\beta}{\mu} > \frac{r}{\alpha}$, the nullclines intersect at a nonzero point, leading to an endemic equilibrium.

Stability Analysis

The stability of the equilibrium points can be determined by evaluating the Jacobian matrix:

$$J(h,m) = \begin{bmatrix} -\alpha m - r & \alpha(1-h) \\ \beta(1-m) & -\beta h - \mu \end{bmatrix}.$$

At $(h,m) = (0,0)$, the eigenvalues of the Jacobian are negative if $\frac{\beta}{\mu} < \frac{r}{\alpha}$, confirming stability.
At the endemic equilibrium (h^*, m^*), stability depends on the condition:

$$\alpha\beta - \mu r > 0.$$

The parameters influencing the system include:

- a: the biting rate, typically $0.2 - 0.5$ bites/day,
- b, c: fractions of bites leading to infection, often ~ 0.5,
- r: recovery rate, $0.01 - 0.05$ per day,
- μ: mosquito mortality rate, $0.05 - 0.5$ per day,
- M_T/H_T: population ratio, typically ~ 2.

The condition for endemicity, rewritten as:

$$\frac{acM_T}{\mu} \cdot \frac{abH_T}{r} > 1,$$

emphasizes the importance of reducing mosquito populations (M_T) and interrupting transmission (b, c) to control malaria.

Lecture 10: Numerical Methods for Differential Equations

Differential equations are fundamental tools in modeling various phenomena in physics, engineering, biology, economics, and many other fields. However, many differential equations cannot be solved analytically or their analytical solutions are too complex for practical use. In such cases, numerical methods provide approximate solutions that are sufficiently accurate for practical purposes.

In this chapter, we delve into the numerical techniques for solving ordinary differential equations (ODEs). We focus on initial value problems (IVPs) of the form:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \tag{10.1}$$

where $y(t)$ is the unknown function, $f(t, y)$ is a given function, and y_0 is the initial condition at time t_0.

10.1 Picard Iteration: Successive Approximations

Picard iteration is an iterative technique derived from fixed-point iteration. It provides a method to approximate the solution of an initial value problem (IVP):

$$x' = f(t, x), \quad x(t_0) = x_0. \tag{10.2}$$

The approach reformulates the differential equation into an equivalent integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{10.3}$$

Picard iteration generates a sequence of approximations $x_0(t), x_1(t), x_2(t), \ldots$, which converge to the exact solution under certain conditions.

The iteration formula is:

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_k(s))\, ds, \quad k = 0, 1, 2, \ldots, \tag{10.4}$$

where the initial guess is typically:

$$x_0(t) = x_0. \tag{10.5}$$

Example

Consider the IVP:

$$x' = 2t(1+x), \quad x(0) = 0. \tag{10.6}$$

The equivalent integral equation is:

$$x(t) = \int_0^t 2s(1+x(s))\,ds. \tag{10.7}$$

- **First Iteration ($x_0(t)$):** Start with $x_0(t) = 0$:

$$x_1(t) = \int_0^t 2s(1+0)\,ds = t^2.$$

- **Second Iteration ($x_1(t)$):** Substitute $x_1(t)$:

$$x_2(t) = \int_0^t 2s(1+s^2)\,ds = t^2 + \frac{1}{2}t^4.$$

- **Third Iteration ($x_2(t)$):** Substitute $x_2(t)$:

$$x_3(t) = \int_0^t 2s\left(1+s^2+\frac{1}{2}s^4\right)\,ds = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6.$$

The process generates a sequence of partial sums of the Taylor series of the exact solution:

$$x(t) = e^{t^2} - 1.$$

The method converges if $f(t,x)$ is smooth, with continuous partial derivatives, in a region containing (t_0, x_0). While Picard iteration is often used for theoretical purposes, it is less practical for numerical computations.

10.2 The Euler Method

Euler's method is a simple numerical approximation for solving IVPs:

$$x' = f(t,x), \quad x(t_0) = x_0. \tag{10.8}$$

Derivation

Divide the interval $[t_0, T]$ into N subintervals of equal width $h = \frac{T-t_0}{N}$. The recursive formula is:

$$x_{n+1} = x_n + hf(t_n, x_n), \quad n = 0, 1, 2, \ldots. \tag{10.9}$$

This formula approximates the solution by replacing the derivative with a finite difference.

Example 6.3: Approximating a Solution

Consider the IVP:

$$x' = t - x, \quad x(0) = 1. \tag{10.10}$$

Solve this using $h = 0.25$ over $[0, 2]$. The exact solution is:

$$x(t) = t - 1 + 2e^{-t}.$$

- **Step-by-Step Calculations:**

$$x_0 = 1, \quad x_1 = x_0 + h(t_0 - x_0) = 1 + 0.25(0 - 1) = 0.75.$$

$$x_2 = x_1 + h(t_1 - x_1) = 0.75 + 0.25(0.25 - 0.75) = 0.625.$$

Continue iteratively for $x_3, x_4, \ldots$.

Error Analysis

The local truncation error (LTE) is $\mathcal{O}(h^2)$, while the global truncation error (GTE) is $\mathcal{O}(h)$.

6.2.2 The Modified Euler Method (Heun's Method)

Heun's method improves Euler's method by averaging the slopes at the beginning and end of each interval:

$$x_{n+1} = x_n + \frac{h}{2}\left[f(t_n, x_n) + f(t_{n+1}, x_n + hf(t_n, x_n))\right]. \tag{10.11}$$

This reduces the global error to $\mathcal{O}(h^2)$.

6.2.3 The Runge–Kutta Method

The fourth-order Runge–Kutta method provides high accuracy. The update formula is:

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{10.12}$$

where:

$$k_1 = f(t_n, x_n), \quad k_2 = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_1\right),$$

$$k_3 = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_2\right), \quad k_4 = f(t_n + h, x_n + hk_3).$$

Example: Solving an IVP numerically

Consider the initial value problem:

$$\frac{dy}{dt} = y - t^2 + 1, \quad y(0) = 0. \tag{10.13}$$

The exact solution is:

$$y(t) = (t + 1)^2 - 0.5e^t.$$

We aim to approximate $y(t)$ over the interval $[0, 0.5]$ using Euler's method with step size $h = 0.1$. The iterations proceed as follows:

$$\begin{aligned} t_0 &= 0, \quad y_0 = 0, \\ t_{n+1} &= t_n + h, \\ y_{n+1} &= y_n + h\left(y_n - t_n^2 + 1\right). \end{aligned}$$

The results of the computations are summarized in the table below, alongside the exact solution and the error at each step:

t	Euler y	True y	Error
0.0	0.000	0.500	0.500
0.1	0.100	0.657	0.557
0.2	0.209	0.829	0.620
0.3	0.326	1.015	0.689
0.4	0.449	1.214	0.765

Table 10.1: Euler's Method Approximation for $\frac{dy}{dt} = y - t^2 + 1$

To visualize the approximation, the plot below shows the Euler method results compared to the exact solution:

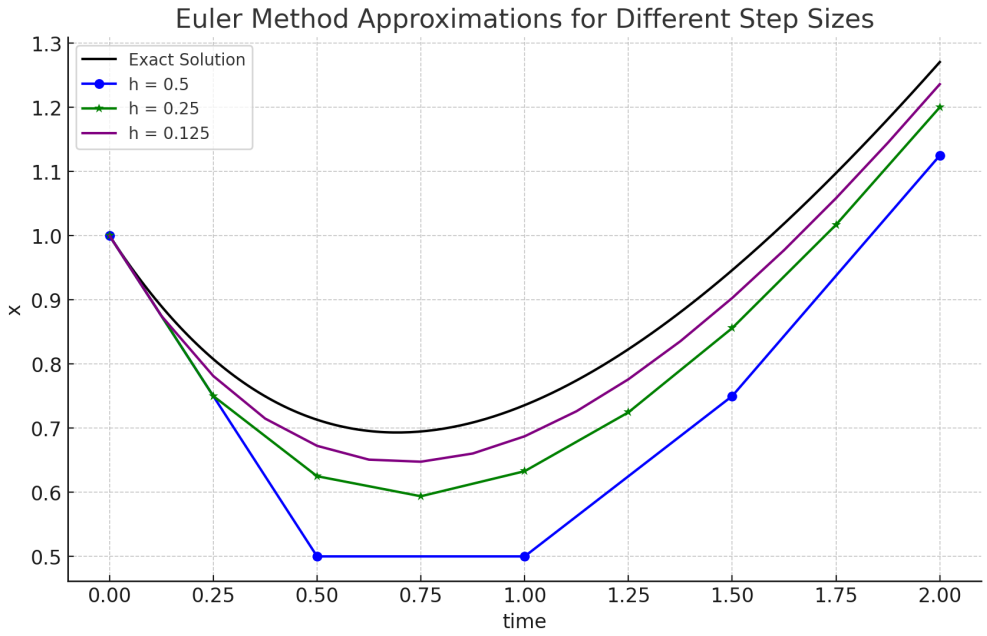


Figure 10.1: Comparison of Euler's Method Approximation and True Solution

Euler's method provides a simple and intuitive way to approximate solutions to ODEs. However, as seen in the example, the method introduces an error that depends on the step size h. Larger h values lead to greater discrepancies from the true solution. While Euler's method is computationally inexpensive, its accuracy can be limited for problems requiring high precision or for stiff equations.

To improve the accuracy of Euler's method, we can use Heun's method, which is a predictor-corrector scheme. It estimates the slope at the beginning and end of the interval and averages them.

The update formula is:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right]. \tag{10.14}$$

Lecture 11: Stochastic Models

Stochastic models incorporate randomness and uncertainty into the modeling process. They are essential for accurately describing systems where random events play a significant role, especially in small populations. Unlike deterministic models, which predict exact outcomes given initial conditions, stochastic models acknowledge the inherent unpredictability in systems due to random fluctuations.

A fundamental concept in stochastic processes is the *Discrete-State Continuous-Time Markov Process*. In such processes, the system transitions between discrete states at random times, and the future state depends only on the current state, not on the sequence of events that preceded it. This property is known as the *Markov property*.

The evolution of the probability distribution $P(\mathbf{n}, t)$ of being in state $\mathbf{n}$ at time t is governed by the *Master Equation*:

$$\frac{dP(\mathbf{n}, t)}{dt} = \sum_{\mathbf{n}' \neq \mathbf{n}} \left[W(\mathbf{n}' \to \mathbf{n}) P(\mathbf{n}', t) - W(\mathbf{n} \to \mathbf{n}') P(\mathbf{n}, t) \right], \tag{11.1}$$

where $W(\mathbf{n} \to \mathbf{n}')$ is the transition rate from state $\mathbf{n}$ to state $\mathbf{n}'$. This equation consists of two main terms: the *gain term*, which accounts for the probability flow into state $\mathbf{n}$ from all other states, and the *loss term*, which accounts for the probability flow out of state $\mathbf{n}$ to all other states. The Master Equation ensures the conservation of probability, meaning the total probability across all states remains equal to 1 at all times.

To derive the Master Equation, consider an infinitesimal time interval $[t, t+dt)$. The change in probability $P(\mathbf{n}, t)$ over dt is due to transitions into and out of state $\mathbf{n}$. The probability of transitioning from $\mathbf{n}'$ to $\mathbf{n}$ in time dt is $W(\mathbf{n}' \to \mathbf{n}) P(\mathbf{n}', t) dt$, and the probability of leaving $\mathbf{n}$ to any other state is $W(\mathbf{n} \to \mathbf{n}') P(\mathbf{n}, t) dt$. Summing over all possible transitions and taking the limit as $dt \to 0$, we obtain the Master Equation.

11.1 Birth-Death Processes

As a concrete example, consider the *birth-death process*, a fundamental model in population dynamics representing the growth and decline of a population. In this model, the population can increase by one (birth) or decrease by one (death). Let N denote the population size.

The transition rates are defined as:

$$W(N \to N+1) = bN, \quad \text{(Birth rate)} \tag{11.2}$$

$$W(N \to N-1) = dN, \quad \text{(Death rate)} \tag{11.3}$$

where b is the per capita birth rate, and d is the per capita death rate.

The Master Equation for the birth-death process becomes:

$$\frac{dP(N, t)}{dt} = b(N-1)P(N-1, t) + d(N+1)P(N+1, t) - (b+d)NP(N, t). \tag{11.4}$$

To analyze this equation, we can compute the mean and variance of the population size. The mean population size $\mathbb{E}[N]$ evolves according to:

$$\frac{d\mathbb{E}[N]}{dt} = (b-d)\mathbb{E}[N]. \tag{11.5}$$

This differential equation describes exponential growth or decay of the mean population size, depending on whether $b > d$ or $b < d$.

The variance $\mathrm{Var}(N)$ evolves as:

$$\frac{d\mathrm{Var}(N)}{dt} = (b+d)\mathbb{E}[N]. \tag{11.6}$$

This shows that the variability in population size increases over time, influenced by both birth and death rates.

In the special case where $b = d = \lambda$, the steady-state solution of the Master Equation can be found by applying the detailed balance condition:

$$b(N-1)P_{\mathrm{ss}}(N-1) = dNP_{\mathrm{ss}}(N). \tag{11.7}$$

This recursion relation leads to a Poisson distribution for the steady-state probabilities:

$$P_{\mathrm{ss}}(N) = \frac{e^{-\lambda}\lambda^N}{N!}. \tag{11.8}$$

11.2 Predator-Prey Models

In the stochastic version, populations are discrete, and interactions occur randomly with certain probabilities. The possible events and their rates are:

$$\text{Prey birth:} \quad (x,y) \to (x+1,y) \quad \text{at rate } \alpha x, \tag{11.9}$$

$$\text{Predation:} \quad (x,y) \to (x-1,y+1) \quad \text{at rate } \beta xy, \tag{11.10}$$

$$\text{Predator death:} \quad (x,y) \to (x,y-1) \quad \text{at rate } \gamma y. \tag{11.11}$$

The Master Equation for this system is complex due to the two-dimensional state space but can be simulated using the Gillespie Algorithm by extending it to handle multiple types of events.

Stochastic simulations of predator-prey models reveal phenomena like extinction of species, irregular population cycles, and noise-induced transitions, which are not captured by deterministic models.

11.3 SIR Epidemiological Models

In the stochastic SIR model, the events are:

$$\text{Infection:} \quad (S,I,R) \to (S-1,I+1,R) \quad \text{at rate } \beta SI, \tag{11.12}$$

$$\text{Recovery:} \quad (S,I,R) \to (S,I-1,R+1) \quad \text{at rate } \gamma I. \tag{11.13}$$

The stochastic model accounts for random variations in disease transmission and recovery, which can lead to different outbreak sizes and the possibility of disease extinction even when $R_0 > 1$.

Connections Between Deterministic and Stochastic Models

Understanding the relationship between deterministic and stochastic models is crucial. In large populations, stochastic effects average out, and the system behaves deterministically. The deterministic models often represent the *mean-field approximation* of the stochastic models.

For example, the expected population size in the stochastic birth-death process satisfies:

$$\frac{d\mathbb{E}[N]}{dt} = (b-d)\mathbb{E}[N], \tag{11.14}$$

which matches the deterministic equation.

Moreover, the Master Equation can be approximated by the *Fokker-Planck Equation* in the limit of large populations and continuous variables. The Fokker-Planck Equation describes the time evolution of the probability density function $P(\mathbf{x},t)$ and is closely related to *Stochastic Differential Equations* (SDEs), such as the Langevin equation.

Lecture 12: Interdisciplinary approaches

12.1 Game theory

Game Theory is a mathematical framework for analyzing strategic interactions among rational decision-makers, referred to as players. Each player selects a strategy from a set of possible actions, and the outcome for each player depends on the combination of strategies chosen by all players involved. The fundamental goal in Game Theory is to determine the optimal strategies that players should adopt to maximize their respective payoffs, often leading to the concept of Nash Equilibrium— a state where no player can benefit by unilaterally changing their strategy.

Evolutionary Game Theory and Replicator Dynamics

Extending classical Game Theory, Evolutionary Game Theory incorporates the dynamics of strategy evolution within populations. Instead of assuming that players are perfectly rational, this approach considers that strategies proliferate based on their success or fitness relative to other strategies in the population. The change in frequency of a particular strategy over time is governed by the Replicator Dynamics, which can be mathematically expressed as:

$$\frac{dx_i}{dt} = x_i \left(f_i - \bar{f}\right)$$

Here, x_i denotes the frequency of strategy i within the population, f_i represents the fitness (or payoff) of strategy i, and $\bar{f}$ is the average fitness of the entire population. This equation implies that strategies with fitness above the average increase in frequency, while those below the average decline.

Connecting Replicator Dynamics to Lotka-Volterra Equations

The mathematical structure of Replicator Dynamics bears a strong resemblance to the Lotka-Volterra Equations, which model predator-prey interactions in biological ecosystems. To elucidate this connection, consider interpreting strategies in Game Theory as species in an ecological model. Specifically, let us map one strategy to a prey species and another to a predator species.

Suppose we have two strategies: Strategy A (analogous to prey) and Strategy B (analogous to predators). The fitness functions for these strategies can be defined as:

$$f_A = \alpha - \beta y \quad \text{and} \quad f_B = \delta x - \gamma$$

In these expressions, α represents the intrinsic fitness of Strategy A in the absence of Strategy B, β captures the negative impact of Strategy B on Strategy A's fitness, δ quantifies the positive impact of Strategy A on Strategy B's fitness, and γ denotes the intrinsic fitness cost of Strategy B.

Substituting these fitness functions into the Replicator Equations yields:

$$\frac{dx}{dt} = x(\alpha - \beta y - \bar{f}) \quad \text{and} \quad \frac{dy}{dt} = y(\delta x - \gamma - \bar{f})$$

Calculating the average fitness $\bar{f}$:

$$\bar{f} = x f_A + y f_B = \alpha x - \beta xy + \delta xy - \gamma y$$

Substituting $\bar{f}$ back into the Replicator Equations and simplifying, we obtain:

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$

These equations are precisely the Lotka-Volterra Equations:

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$

This mapping demonstrates that the dynamics of strategy frequencies in Evolutionary Game Theory can be directly translated into the dynamics of interacting species populations in ecological models.

Illustrative Example: The Hawk-Dove Game

To concretize the connection, consider the Hawk-Dove Game, a classic model in Evolutionary Game Theory that represents conflict and sharing behaviors.

Game Setup

There are two strategies:

- **Hawk**: An aggressive strategy that always fights over resources.
- **Dove**: A passive strategy that shares resources or retreats if confronted by a Hawk.

The payoff matrix is defined as:

	Hawk	Dove
Hawk	$\frac{V-C}{2}$	V
Dove	0	$\frac{V}{2}$

where V is the value of the resource and C is the cost of fighting ($C > V$).

Replicator Equations for the Hawk-Dove Game

Let x be the frequency of Hawks and $y = 1 - x$ be the frequency of Doves in the population. The fitness of each strategy is:

$$f_H = \frac{V-C}{2}x + Vy \quad \text{and} \quad f_D = 0 \cdot x + \frac{V}{2}y$$

The average fitness $\bar{f}$ is:

$$\bar{f} = xf_H + yf_D = \frac{V-C}{2}x^2 + Vxy + \frac{V}{2}y^2$$

Substituting these into the Replicator Equations:

$$\frac{dx}{dt} = x\left(\frac{V-C}{2}x + Vy - \bar{f}\right)$$
$$\frac{dy}{dt} = y\left(\frac{V}{2}y - \bar{f}\right)$$

Through simplification, these equations can be mapped to the Lotka-Volterra form, illustrating cyclical dynamics akin to predator-prey interactions.

This interdisciplinary connection allows for the application of ecological modeling techniques to strategic interactions and vice versa. For instance, understanding the stability and oscillatory behavior of populations in ecology can inform strategies in competitive environments such as economics or social systems. Conversely, insights from strategic behavior can enhance models of biological interactions, providing a richer understanding of both fields.

12.2 Neural Ordinary Differential Equations

Neural Ordinary Differential Equations (Neural ODEs) provide a continuous formulation of deep learning models, generalizing discrete layer-based architectures such as residual networks. In traditional models, transformations are applied sequentially in layers, with residual networks described by the iterative rule

$$h_{t+1} = h_t + f(h_t, \theta_t), \tag{12.1}$$

where $h_t \in \mathbb{R}^D$ is the hidden state at layer t, and $f(h_t, \theta_t)$ is a learned transformation. This process is analogous to the Euler discretization of an ordinary differential equation (ODE), suggesting that the layer-wise updates approximate the dynamics of a continuous system.

Neural ODEs formalize this idea by defining the hidden state dynamics as a differential equation:

$$\frac{dh(t)}{dt} = f(h(t), t, \theta), \tag{12.2}$$

where $h(t) \in \mathbb{R}^D$ is the hidden state at continuous time t, and $f(h(t), t, \theta)$ is a neural network parameterized by θ. Given an initial condition $h(0)$, the solution to this initial value problem at time T represents the output of the model:

$$h(T) = h(0) + \int_0^T f(h(t), t, \theta)\, dt. \tag{12.3}$$

This solution is computed using numerical ODE solvers, which adaptively evaluate the dynamics to achieve a desired accuracy.

One of the key innovations in Neural ODEs is their approach to backpropagation. Traditional neural networks store intermediate activations during the forward pass to compute gradients during the backward pass. In contrast, Neural ODEs leverage the *adjoint sensitivity method*, which avoids this memory-intensive storage. Let $L(h(T))$ be a scalar loss function. The gradient of the loss with respect to the hidden state $h(t)$ at time t is called the adjoint $a(t)$, defined as

$$a(t) = \frac{\partial L}{\partial h(t)}. \tag{12.4}$$

The adjoint evolves according to another ODE, derived via the chain rule:

$$\frac{da(t)}{dt} = -a(t)^\top \frac{\partial f(h(t), t, \theta)}{\partial h(t)}. \tag{12.5}$$

To compute the gradients with respect to the parameters θ, an additional integral is evaluated:

$$\frac{\partial L}{\partial \theta} = -\int_0^T a(t)^\top \frac{\partial f(h(t), t, \theta)}{\partial \theta}\, dt. \tag{12.6}$$

Both the adjoint dynamics and the parameter gradients are computed efficiently by solving the corresponding ODEs backward in time, resulting in constant memory usage during training.

Neural ODEs exhibit several properties that distinguish them from traditional architectures. The use of numerical solvers allows the model to adapt the number of function evaluations (NFE) based on the complexity of the input and the required precision. This adaptability enables a trade-off between computational cost and accuracy. Moreover, the depth of the model is no longer fixed; instead, the effective depth is determined by the solver, which interprets the dynamics of the system.

One of the most striking applications of Neural ODEs is in *Continuous Normalizing Flows (CNFs)*, a generative modeling framework. Normalizing flows rely on the change-of-variables formula to compute the density of a transformed random variable:

$$\log p(z_1) = \log p(z_0) - \log \left| \det \frac{\partial f}{\partial z_0} \right|. \tag{12.7}$$

In CNFs, this is extended to continuous transformations, where the change in log-density follows the differential equation:

$$\frac{\partial \log p(z(t))}{\partial t} = -\mathrm{tr}\left(\frac{\partial f(z(t), t)}{\partial z(t)} \right). \tag{12.8}$$

This formulation replaces the costly computation of the determinant with a trace operation, making CNFs highly efficient for high-dimensional data. The resulting density model is both scalable and invertible, enabling tasks such as density estimation and sampling.

In addition to generative modeling, Neural ODEs excel in modeling irregular time-series data. By defining the dynamics of a latent trajectory as an ODE, Neural ODEs naturally handle observations at arbitrary time points. Given a latent state $z(t)$ at time t, the evolution of the state is governed by

$$\frac{dz(t)}{dt} = f(z(t), t, \theta). \tag{12.9}$$

Observations $x(t_i)$ are generated from the latent state via a decoder:

$$x(t_i) \sim p(x|z(t_i), \theta_x). \tag{12.10}$$

This framework has been used to reconstruct and extrapolate trajectories, outperforming recurrent neural networks in handling irregularly sampled data.

Despite their strengths, Neural ODEs pose unique challenges. The reliance on numerical solvers introduces computational overhead, and the choice of solver parameters, such as error tolerances, significantly impacts performance. Furthermore, the continuous formulation complicates standard training practices, such as mini-batching, as each batch element may require a different number of evaluations.

Neural ODEs represent a profound shift in how neural networks are conceptualized, offering a mathematically elegant and computationally efficient framework for continuous transformations. By bridging the gap between dynamical systems and deep learning, they open new avenues for research and application in both scientific and real-world domains.