Numerical Computing

Lecture 12: Partial Differential Equations

Francisco Richter Mendoza

Università della Svizzera Italiana
Faculty of Informatics, Lugano, Switzerland

Lecture Overview

- **Classification**: elliptic, parabolic, hyperbolic
- **Finite Differences**: heat and wave discretizations
- **Stability**: von Neumann, CFL
- **Elliptic Solvers**: Poisson stencil, iterative methods
- **FEM and Spectral**: weak form, accuracy

PDE Landscape

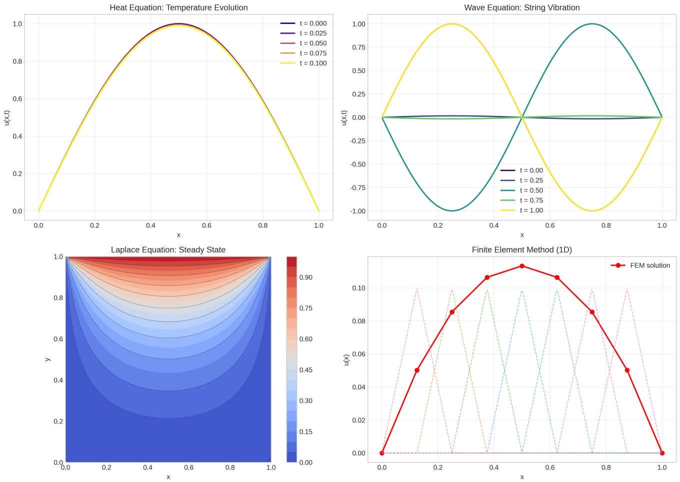

Figure: Elliptic vs Parabolic vs Hyperbolic: data and phenomena.

Classification of Second-Order Linear PDEs

General form

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

Discriminant $\Delta = B^2 - 4AC$:

- Elliptic ($\Delta < 0$) Parabolic ($\Delta = 0$) Hyperbolic ($\Delta > 0$)

Heat Equation: Forward/Backward Euler

Forward Euler

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \alpha \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}; \text{ stable if } r \leq 1/2.$$

Backward Euler

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \alpha \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{(\Delta x)^2}; \text{ A-stable.}$$

Stability: Heat (Forward Euler)

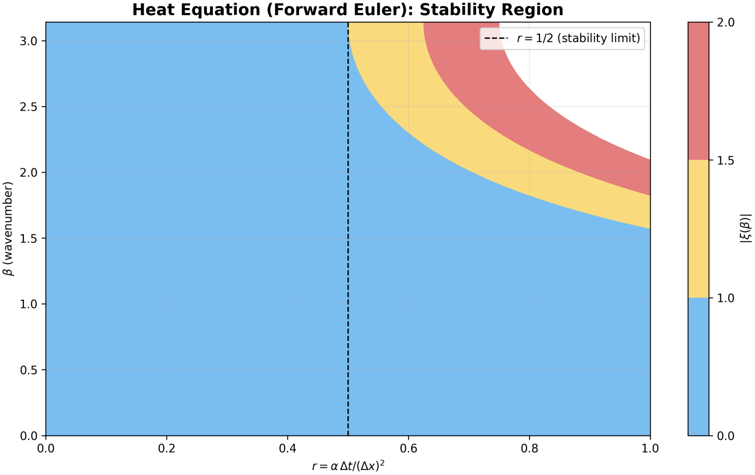

Figure: Amplification factor $|\xi(\beta)|$ vs r and wavenumber; stable if $r \leq 1/2$.

Wave Equation: Leapfrog

Scheme

$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{(\Delta t)^2} = c^2 \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}.$$

CFL: $c \, \Delta t / \Delta x \leq 1$.

CFL Stability Diagram

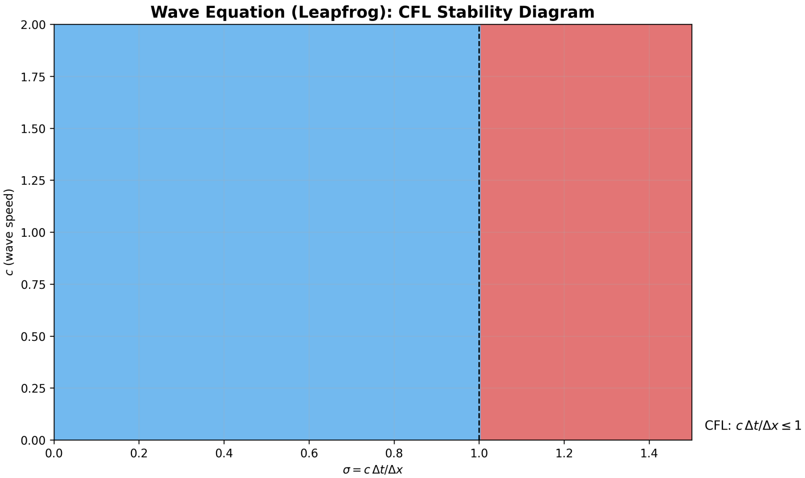

Figure: Stable region for leapfrog: $\sigma = c\,\Delta t/\Delta x \leq 1$.

Poisson: Five-Point Stencil

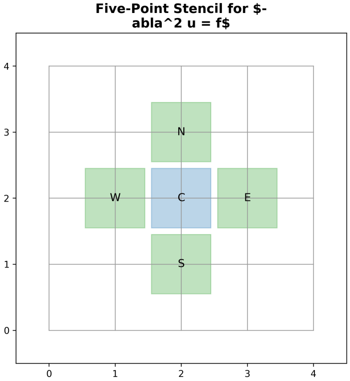

Figure: Stencil layout for $-\nabla^2 u = f$ on a uniform grid.

Iterative Solvers for Elliptic Problems

- Gauss–Seidel; SOR with relaxation ω
- Multigrid: smooth, restrict, coarse solve, prolongate

Finite Element: Weak Form

Poisson

Find $u \in H_0^1(\Omega)$: $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx$ for all $v \in H_0^1(\Omega)$.

Discretization Families

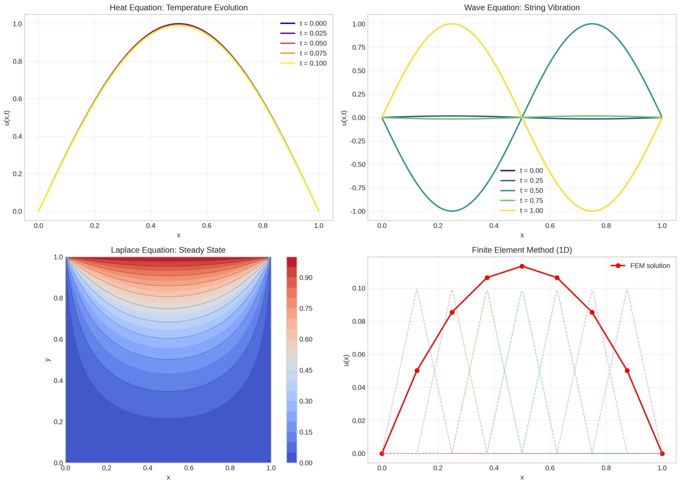

Figure: Families at a glance; see notes for detailed comparison (FD/FEM/Spectral).

Spectral Methods

- Fourier (periodic): exponential accuracy for smooth solutions
- Chebyshev (nonperiodic): global polynomials; clustering at boundaries

Key Takeaways

- **PDE classes** dictate admissible discretizations and data
- **Stability** (von Neumann, CFL) is central
- **Elliptic solvers**: iterative and multigrid options
- **FEM/Spectral**: weak formulations and accuracy