Numerical Computing

Lecture 11: Ordinary Differential Equations

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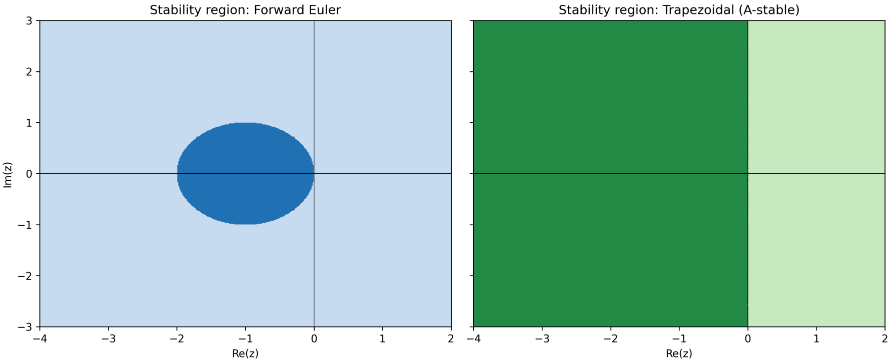
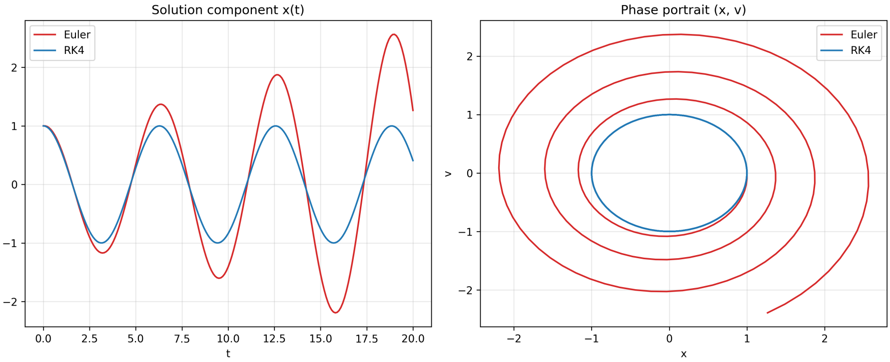
Reading the Stability Plot



For the test equation $y' = \lambda y$ with $z = h\lambda$:

- Euler: stable if z lies in the unit disk centered at -1 (small h if $\Re\lambda \ll 0$)
- Trapezoidal: A-stable (entire left half-plane). Choose h by accuracy, not stability

Why Does Euler Drift?



Harmonic oscillator conserves energy; explicit Euler perturbs energy (spiral). Higher-order RK4 better preserves invariants over moderate times.

Worked Example: Euler vs RK2 on $y' = -y$, $y(0) = 1$, $h = 0.5$

Exact: $y(t) = e^{-t} \quad \Rightarrow \quad y(0.5) \approx 0.6065$, $y(1.0) \approx 0.3679$.

Euler: $y_1 = 1 + 0.5(-1) = 0.5, \quad y_2 = 0.5 + 0.5(-0.5) = 0.25$.

Heun (RK2): predictor $\tilde{y}_1 = 1 + 0.5(-1) = 0.5, \quad$ corrector $y_1 = 1 + \frac{0.5}{2}(-1 - 0.5) = 0.625$.
Next step: $\tilde{y}_2 = 0.625 + 0.5(-0.625) = 0.3125$,
$y_2 = 0.625 + \frac{0.5}{2}(-0.625 - 0.3125) = 0.390625$.

RK2 tracks the exact values better than Euler with the same h.

Lecture Overview

- **IVPs**: existence/uniqueness; Lipschitz; conditioning
- **One-step methods**: Euler (fwd/backward), Heun, RK2, RK4
- **Multistep**: Adams, BDF (stiffness-oriented)
- **Stability**: regions; A-stability; stiffness
- **Adaptive methods**: embedded RK; PI step controller

Initial Value Problems (IVPs)

Problem

Given $y'(t) = f(t, y(t))$ on $[t_0, T]$ with $y(t_0) = y_0$, approximate $y(t)$ at grid points $t_n = t_0 + nh$.

Example (harmonic oscillator): $y' = \begin{pmatrix} v \\ -y \end{pmatrix}$, $y(0) = (1, 0)^T$. We seek $y(t_n)$ without solving analytically.

Existence and Uniqueness

Picard–Lindelöf

If f is Lipschitz in y and continuous in t near (t_0, y_0), then a unique local solution exists, obtained by Picard iteration.

Continuous dependence (Grönwall): if $\|f(t, y_1) - f(t, y_2)\| \leq L\|y_1 - y_2\|$ then

$$\|y(t) - z(t)\| \leq e^{L(t-t_0)} \|y_0 - z_0\|.$$

Lipschitz Continuity and Conditioning

Lipschitz

$\|f(t, y_1) - f(t, y_2)\| \leq L \|y_1 - y_2\|$ controls sensitivity to perturbations.

Implication: numerical error grows like $\lesssim e^{L(t-t_0)}$; large L suggests smaller h or higher-order methods.

Euler vs RK4

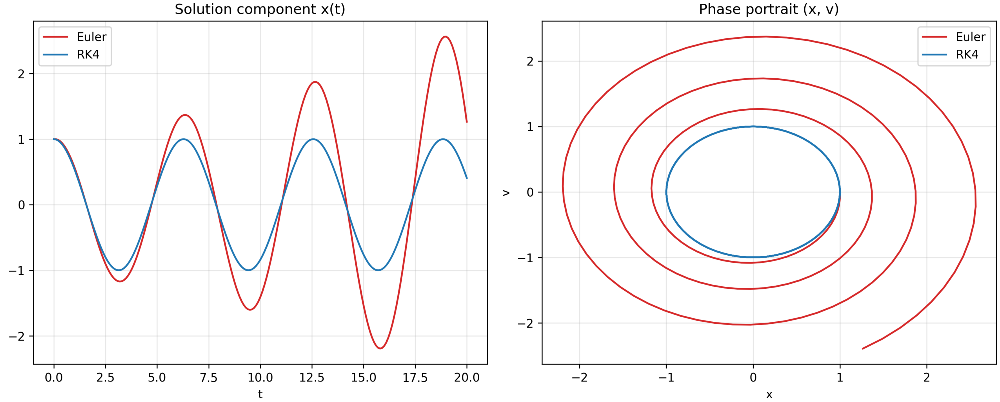


Figure: Euler drift vs RK4 accuracy on harmonic oscillator; phase-portrait comparison.

Forward Euler

Method

$y_{n+1} = y_n + hf(t_n, y_n), \quad t_{n+1} = t_n + h.$

Local error (via Taylor):

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2} y''(\xi_n), \quad \Rightarrow \tau = O(h^2).$$

Stability on $y' = \lambda y$: $y_{n+1} = (1 + h\lambda) y_n$ is stable if $|1 + h\lambda| \leq 1$.

Backward Euler (Implicit)

Method

$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$ (solve for y_{n+1}).

Newton solve for $F(y_{n+1}) = y_{n+1} - y_n - hf(t_{n+1}, y_{n+1}) = 0$:

$$y_{n+1}^{(k+1)} = y_{n+1}^{(k)} - \left(I - hJ_f\right)^{-1}\left(y_{n+1}^{(k)} - y_n - hf(t_{n+1}, y_{n+1}^{(k)})\right).$$

A-stable $\Rightarrow$ well-suited for stiff problems.

Heun / RK2 (Predictor–Corrector)

Scheme

$\tilde{y}_{n+1} = y_n + hf(t_n, y_n); \quad y_{n+1} = y_n + \frac{h}{2}\big[f(t_n, y_n) + f(t_{n+1}, \tilde{y}_{n+1})\big].$

Tableau: $\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$ gives order 2.

Runge–Kutta via Butcher Tableau

$$
\begin{array}{c|cccc}
c_1 & & & & \\
c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
\vdots & \vdots & \vdots & \ddots & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} \\
\hline
 & b_1 & b_2 & \cdots & b_s
\end{array}
$$

- Stages: $k_i = f(t_n + c_i h,\ y_n + h \sum_j a_{ij} k_j)$; update: $y_{n+1} = y_n + h \sum_i b_i k_i$

Classical RK4

$$k_1 = f(t_n, y_n), \qquad k_2 = f(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_1),$$

$$k_3 = f(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_2), \qquad k_4 = f(t_n + h, y_n + hk_3),$$

$$y_{n+1} = y_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

- Order 4; workhorse for non-stiff IVPs

Stability Regions

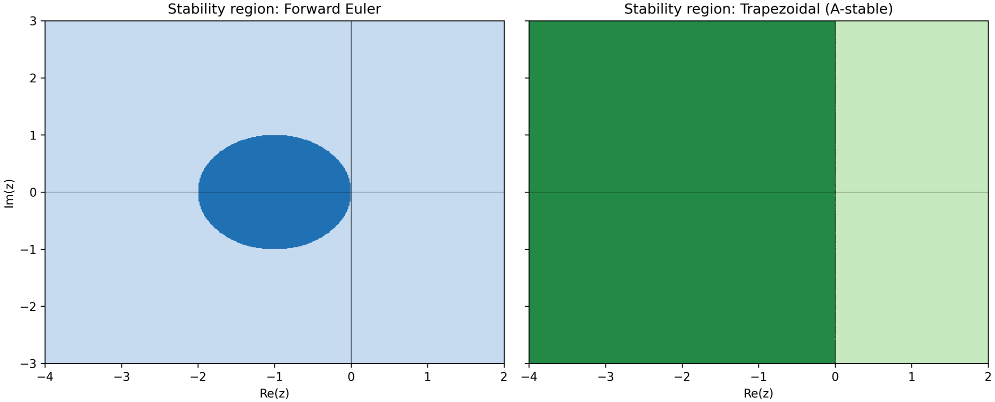


Figure: Absolute stability regions for Forward Euler vs Trapezoidal (A-stable).

Multistep Methods

AB2 (explicit): $y_{n+1} = y_n + \frac{h}{2}\big(3f_n - f_{n-1}\big) \quad$ (order 2).

AM2 (implicit): $y_{n+1} = y_n + \frac{h}{12}\big(5f_{n+1} + 8f_n - f_{n-1}\big)$.

BDF2 (stiff): $\frac{1}{h}\big(\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1}\big) = f_{n+1}$.

Stiff ODE: Stability Matters

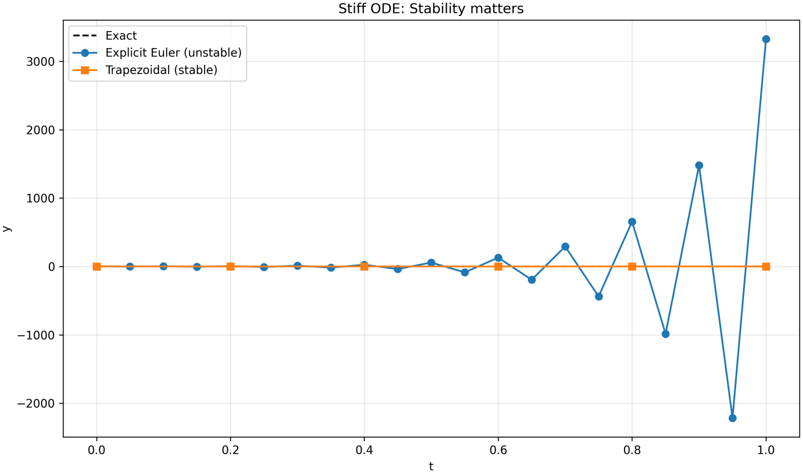


Figure: Explicit Euler instability vs Trapezoidal stability for $y' = -\lambda y$.

What is Stiffness?

Test equation $y' = \lambda y$ with $\Re\lambda \ll 0$. Explicit Euler requires $h \lesssim 2/|\lambda|$ due to $|1 + h\lambda| \leq 1$.

Takeaway: use A-stable/L-stable implicit schemes (e.g., Trapezoidal, BDF) to step with larger h safely.

Embedded RK Error Control

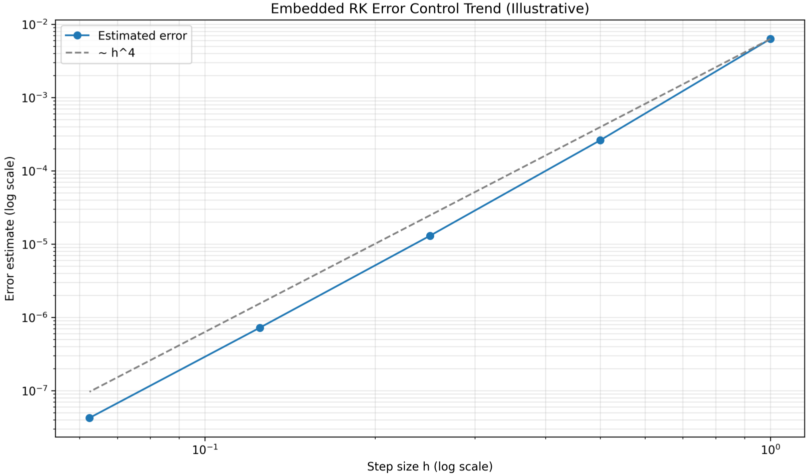


Figure: Error trend with step refinement; reference $\sim h^4$ slope.

Adaptive Step Size Control

PI Controller

$$h_{n+1} = \eta\, h_n \left(\frac{\mathrm{TOL}}{\mathrm{ERR}_n}\right)^{k_P} \left(\frac{\mathrm{ERR}_{n-1}}{\mathrm{ERR}_n}\right)^{k_I}, \text{ typical } k_P \approx 0.7/p,\ k_I \approx 0.4/p.$$

Accept if $\mathrm{ERR}_n \leq \mathrm{TOL}$, else reject and reduce h. Embedded RK (e.g., DOPRI(4,5)) supplies ERR_n.

Systems and Higher-Order ODEs

Systems: $\mathrm{y}' = \mathrm{f}(t, \mathrm{y})$, apply methods componentwise.

Reduction: $y^{(m)} = g(\cdot) \;\Rightarrow\; \mathrm{z}' = \begin{pmatrix} z_2 \\ \vdots \\ g(t, z_1, \ldots, z_m) \end{pmatrix}$ with $z_1 = y$.

Boundary Value Problems (BVPs)

Shooting: choose initial slope s, solve IVP, match $y(T; s)$ to boundary via $\phi(s) = y(T; s) - \beta = 0$.

Finite differences: replace derivatives by stencils on a grid and solve the resulting algebraic system.

Key Takeaways

- **Consistency + Stability => Convergence** (Dahlquist)
- **A-stability** is key for stiff problems
- **Embedded RK** enables adaptive step size control