Numerical Computing

Lecture 9: Interpolation and Approximation

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Lecture Overview

- **Polynomial Interpolation**: Lagrange and Newton forms
- **Error Analysis**: Interpolation error, Runge phenomenon
- **Chebyshev Points**: Optimal nodes on $[-1, 1]$
- **Least Squares Approximation**: Over/under-determined data
- **Conditioning**: Vandermonde matrix and stability

Lagrange Interpolation

Definition

$p(x) = \sum_{i=0}^{n} y_i L_i(x), \quad L_i(x) = \prod_{j \neq i} \dfrac{x - x_j}{x_i - x_j}$

- **Property**: $L_i(x_j) = \delta_{ij}$
- **Uniqueness**: Vandermonde system is nonsingular for distinct nodes

Newton Interpolation

Divided Differences

$$f[x_i, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

Newton Form

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots$$

Advantage: easy to add new nodes incrementally.

Interpolation: Methods and Behavior

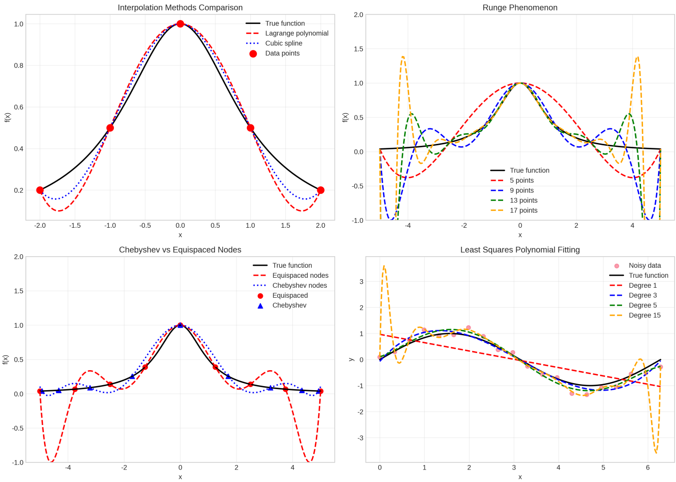

Figure: Polynomial vs. spline interpolation behavior near boundaries and node placement.

Runge Phenomenon

Problem

Equally spaced nodes can cause oscillations for high-degree polynomials.

- Example: $f(x) = 1/(1 + 25x^2)$ on $[-1, 1]$
- **Mitigation**: use Chebyshev nodes

Chebyshev Points

Definition

$$x_k = \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right), \; k = 0, \ldots, n$$

- Minimize the maximum of $\prod_{i=0}^{n} |x - x_i|$ on $[-1, 1]$
- Lower Lebesgue constant: $\Lambda_n = O(\log n)$

Least Squares Approximation

Discrete Problem

Given (x_i, y_i), find c to minimize $\sum_i (y_i - \sum_j c_j \phi_j(x_i))^2$.

- Normal equations or QR factorization
- Choice of basis affects conditioning and stability

Conditioning: Vandermonde Matrix

Issue

Equally spaced nodes lead to ill-conditioned Vandermonde matrices.

Remedies: Chebyshev nodes, orthogonal polynomial bases.

Runge Phenomenon

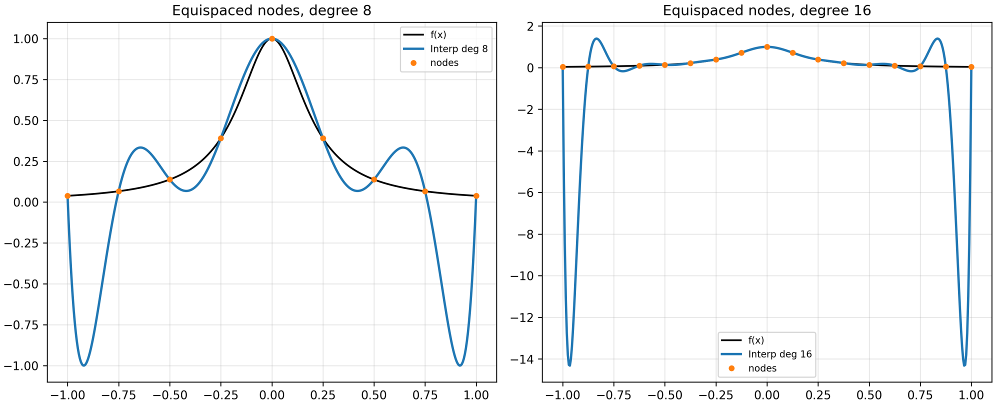

Figure: Oscillations and boundary blow-up for equispaced high-degree interpolation of $f(x) = 1/(1 + 25x^2)$.

Chebyshev vs Equispaced Nodes

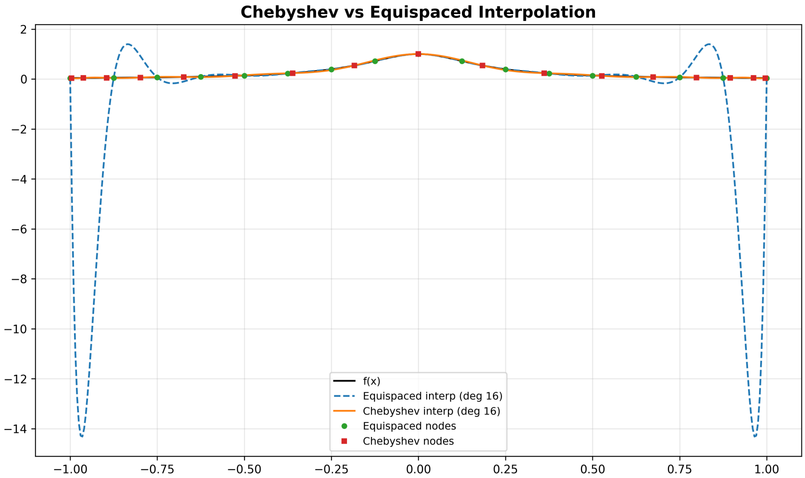

Figure: Chebyshev nodes dramatically improve interpolation accuracy and stability versus equispaced nodes.

Least Squares Polynomial Fits

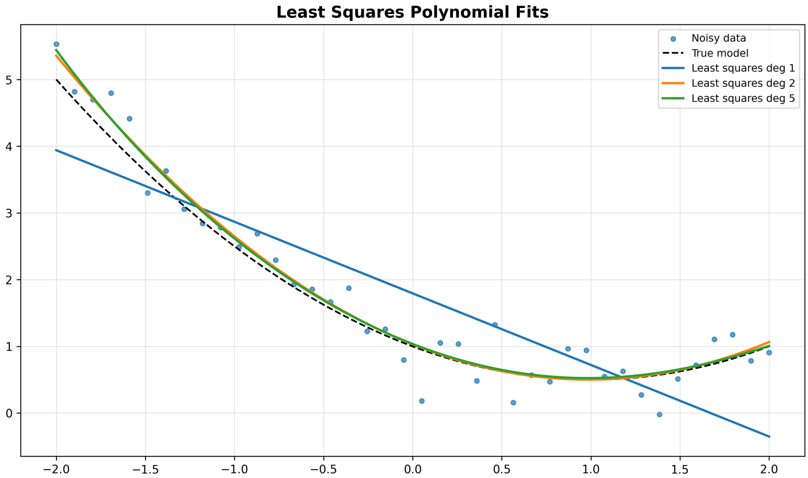

Figure: Least squares fits of increasing degree to noisy data versus the true quadratic model.

Vandermonde Conditioning

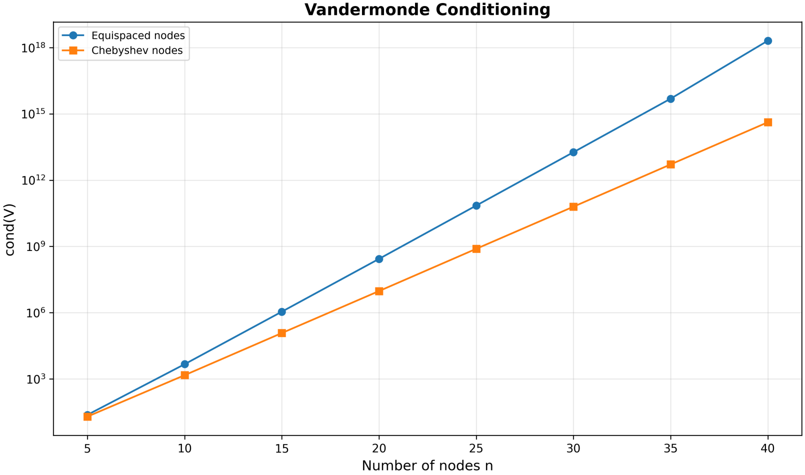

Figure: Condition numbers for Vandermonde matrices using equispaced and Chebyshev nodes as n grows.

Key Takeaways

- **Lagrange/Newton**: equivalent forms; Newton is incremental
- **Runge**: avoid equispaced high-degree interpolation
- **Chebyshev**: optimal nodes for stability
- **Least Squares**: robust fitting in noisy/overdetermined settings
- **Conditioning**: choose nodes and bases wisely